Vertex-Frequency Energy Distributions

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***Abstract*—A vertex-varying spectral content on graphs challenges the assumption of vertex invariance and requires vertex-frequency representations for an adequate analysis. In this letter, we introduce a class of vertex-frequency energy distributions inspired by traditional time-frequency energy distributions. These newly introduced distributions do not use localization windows. Their efficiency in energy concentration is illustrated through examples.**

***Index Terms*—Graph signal processing, Graph spectral representation, Vertex-frequency, Energy distribution.**

I. Introduction

Graphs and graph signal processing have become active research areas in recent years. They represent a novel way to understand classical signal processing techniques, but are also extremely useful in fully describing novel data types such as brain and social networks [1]–[6]. Graphs typically consist of vertices (nodes) and edges (connections between vertices). When using graphs to represent heterogeneous data, the available information can be conveyed in the strength of the edges of the graph or, alternatively, in signals whose values represent data associated with the vertices of the graph.

As in traditional signal processing, graph signal characteristics can be vertex-varying (a loose analogue to time-varying signals [7]–[10]). A recently proposed vertex-frequency analysis relies on Laplacian matrices to establish connections between vertex changes and the spectral content using localization windows [11], [12]. A different line of work has generalized the notion of time stationarity to signals defined on graphs [13], [14], developing windowing and energy spectral estimation schemes for graph-stationary signals [14].

In this letter, we introduce a class of window independent vertex-frequency distributions based on the idea of the Rihaczek energy distribution [7], [10]. These newly proposed distributions are highly localized in the joint vertex-frequency domain and provide a novel way for a systematic introduction of new vertex-frequency distributions.

II. Spectral Decomposition on Graphs

A. Graph Laplacian

Consider an arbitrary signal $\mathbf{x}$ on an undirected graph. The Laplacian of the undirected graph is defined as

$$\mathbf{L} = \mathbf{D} - \mathbf{W}, \tag{1}$$

where the matrix $\mathbf{W}$ coefficients w_{ij} are the weighting values of the edges connecting vertices i and j and $\mathbf{D}$ is a diagonal matrix with elements $d_i = \sum_{j=1}^{N} w_{ij}$. The elements of the Laplacian of a signal $\mathbf{Lx}$ will be denoted by $\mathcal{L}_x(n)$.

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The Laplacian is an operator that can be used to measure the smoothness of a signal defined on the graph via local dissimilarities as [5]

$$\mathbf{x}^T\mathbf{Lx} = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}(x(i)-x(j))^2, \tag{2}$$

where $x(n), n = 1, 2, \ldots, N$, are the signal $\mathbf{x}$ samples.

The Laplacian can also be considered within the basic electric circuit theory (Kirchhoff matrix) when w_{ij} is the conductance of the edge connecting vertices i and j [15]. Then the vertex potential vector $\mathbf{x}$, corresponding to the graph signal, can be related to the vector of external currents $\mathbf{i}_G$ via the Laplacian as $\mathbf{Lx} = \mathbf{i}_G$.

B. Spectral Decomposition

The Laplacian can be written as

$$\mathbf{L} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T, \tag{3}$$

where columns $\mathbf{u}_k$ of the matrix $\mathbf{U}$ are the eigenvectors of $\mathbf{L}$, and $\mathbf{\Lambda}$ is a diagonal matrix with eigenvalues λ_k on the diagonal. Using the eigenvalue decomposition of the Laplacian we can define spectral representation of a signal $\mathbf{x}$ on the graph as

$$\mathbf{X} = \mathbf{U}^T\mathbf{x}. \tag{4}$$

Components of the spectral transform vector $\mathbf{X}$ are $X(\lambda_k) = \mathbf{u}_k^T\mathbf{x}$. This decomposition can be considered as the graph Fourier transform of $\mathbf{x}$ denoted by $\mathrm{GFT}\{\mathbf{x}\} = \mathbf{U}^T\mathbf{x}$.

If the graph is the cycle, with Laplacian of a signal $\mathcal{L}_x(n) = 2x(n) - x(n-1) - x(n+1)$, then the matrix $\mathbf{U}^T$ is the DFT matrix and the classical definition of DFT is recovered [5].

For the basic electric circuits, from $\mathbf{Lx} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T\mathbf{x} = \mathbf{i}_G$ we get $\mathbf{\Lambda}\mathbf{U}^T\mathbf{x} = \mathbf{U}^T\mathbf{i}_G$ or $\mathbf{\Lambda X} = \mathbf{I}_G$, where $\mathbf{I}_G = \mathbf{U}^T\mathbf{i}_G$.

C. Spectral Localization on a Graph

The vertex-frequency analysis deals with the spectral content localization around each vertex. This analysis is an extension of the classical time-frequency analysis to graph signals. Like in the traditional time-frequency analysis, the spectral transform of a signal localized around the considered vertex n is the basic form of the vertex-frequency analysis. We can define a window function $h_n(i)$ based on the vertex distances that will favor closer vertex samples and decrease farther samples (the neighborhood of vertex 1 is shaded/red in Fig. 1) [15]. The corresponding local vertex spectrum (LVS) is then defined as

$$LS_x(n, \lambda_k) = \mathbf{u}_n^T\mathbf{x}_h = \sum_{i=1}^{N} x(i)h_n(i)u_k(i). \tag{5}$$

A vertex-invariant window can be introduced by defining a generalized shift operator within the graph spectrum framework, as shown later in Section III.C [11]. An illustration of a graph, a signal, its spectrum and vertex-frequency representation is presented in Fig. 1. The LVS is calculated according to (16) and (17) using a window function $h_n(i)$ with $H(\lambda_k) = \exp(-2\lambda_k)$.

Like in the classical time-frequency analysis, the vertex-frequency representation is highly dependent on the localization window width. To avoid this effect, energy distributions are defined in the classical time-frequency analysis. In a similar way, vertex-frequency energy distributions can be defined.

III. Vertex-Frequency Distributions

In this section, we will introduce two forms of vertex-frequency distributions. In the first part, we will follow the electric circuit reasoning. In the second part of the section, our analysis will consider the classical definition of energy based on the squared norm-two of the signal.

A. Vertex-Frequency Power in an Electric Circuit

The power in all edges connected to vertex n is equal to the sum of all $p_j(n) = w_{nj}(x(n) - x(j))^2$ over j, where $x(n)$ is the vertex potential and w_{nj} are the edge conductances. The power within the whole network is equal to

$$P = \frac{1}{2}\sum_{n=1}^{N}\sum_{j=1}^{N} p_j(n). \tag{6}$$

The factor of $\frac{1}{2}$ is a result of the fact that all edges are taken twice in the summation over all vertices in the circuit. It can be shown that (6) can be written as $P = \mathbf{x}^T\mathbf{i}_G$ or as

$$P = \mathbf{x}^T(\mathbf{L}\mathbf{x}) = \sum_{n=1}^{N} x(n)\sum_{j=1}^{N} w_{nj}(x(n) - x(j)). \tag{7}$$

Then the total power in the circuit can be defined as

$$P = \sum_{n=1}^{N}\sum_{k=1}^{N}\sum_{j=1}^{N}\frac{1}{2}w_{nj}(x(n) - x(j))X(\lambda_k)(u_k(n) - u_k(j)). \tag{8}$$

It can be represented as a sum over the vertex and frequency indices as

$$P = \sum_{n=1}^{N}\sum_{k=1}^{N} P(n,k), \text{ with} \tag{9}$$

$$P(n,k) = \sum_{j=1}^{N}\frac{1}{2}w_{nj}(x(n) - x(j))X(\lambda_k)(u_k(n) - u_k(j)).$$

The value of $P(n,k)$ can be considered as a vertex-frequency power distribution of the graph signal $x(n)$. The marginal properties of this distribution are:

$$\sum_{n=1}^{N} P(n,k) = \lambda_k|X(\lambda_k)|^2 = X_D^2(\lambda_k) \tag{10}$$

$$\sum_{k=1}^{N} P(n,k) = \sum_{j=1}^{N}\frac{1}{2}w_{nj}(x(n) - x(j))^2 = x_D^2(n), \tag{11}$$

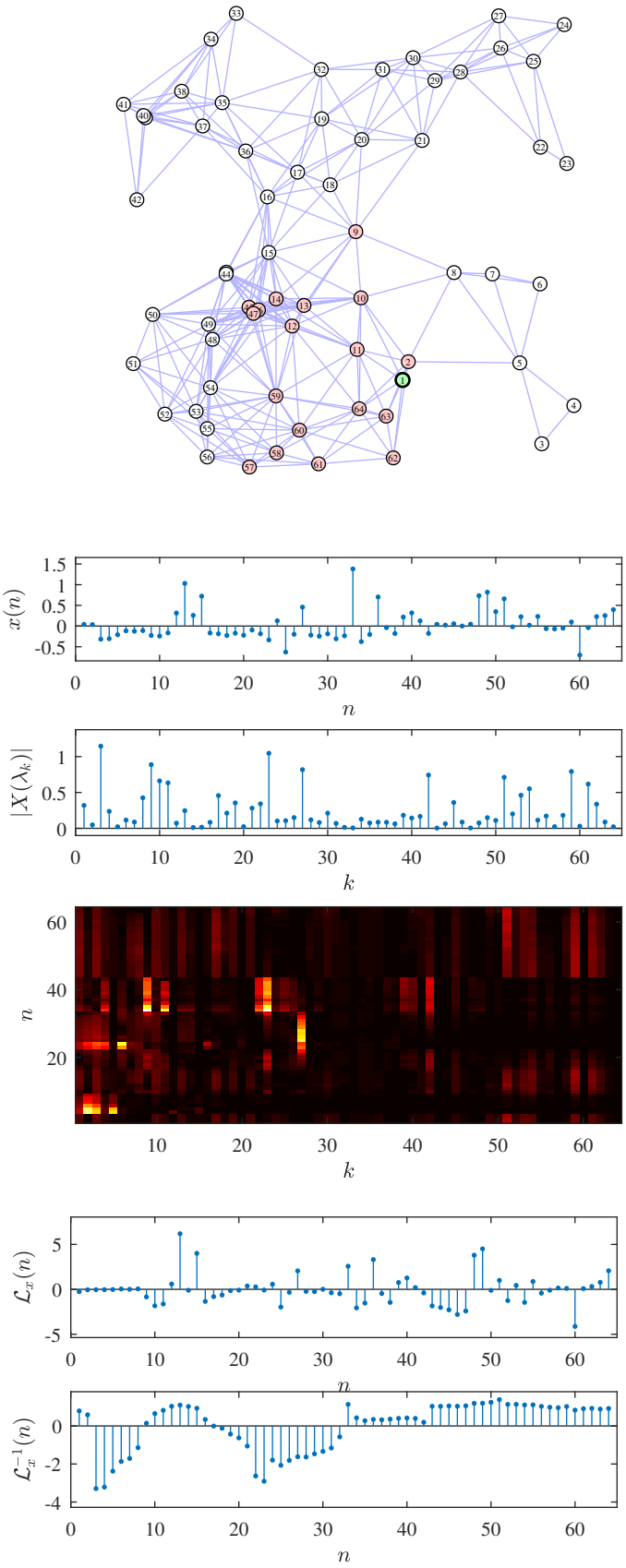

Fig. 1: Graph with a signal $x(n)$, its spectral decomposition $X(\lambda_k)$, local vertex-frequency representation, Laplacian $\mathcal{L}_x(n)$ of the signal $x(n)$ on graph, and its inverse function $\mathcal{L}_x^{-1}(n)$ (with assumed zero-mean value).

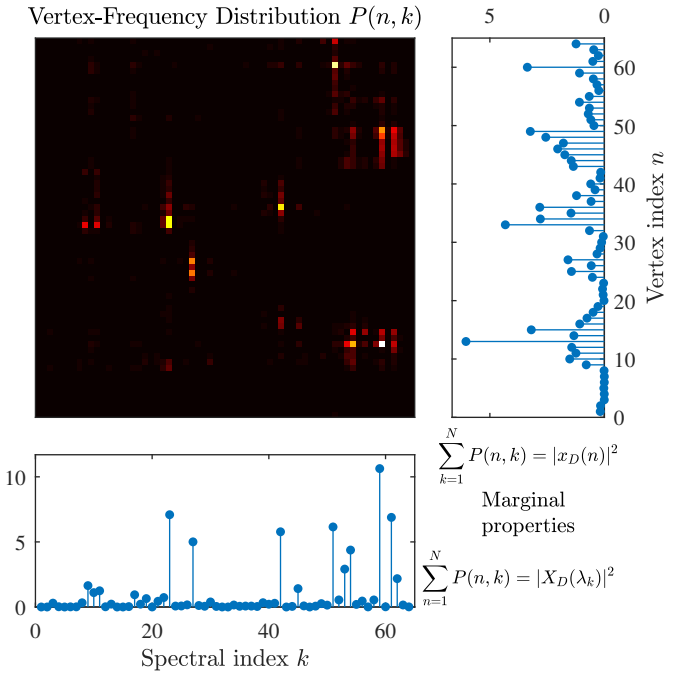

Fig. 2: A vertex-frequency distribution of power in the graph signal from Fig. 1.

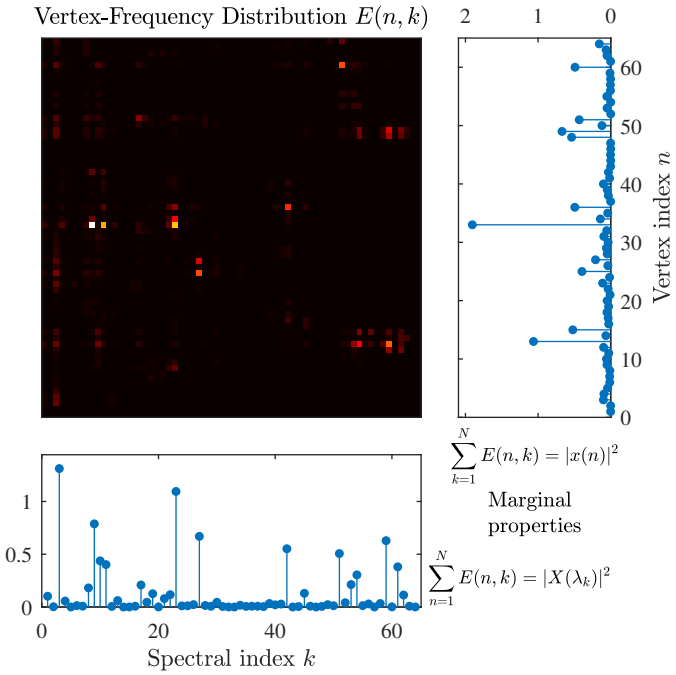

Fig. 3: A vertex-frequency distribution of energy in the graph signal from Fig. 1.

where $x_D^2(n) = x(n)\mathcal{L}_x(n)$.

We have determined that the spectral power is of the form $\lambda_k|X(\lambda_k)|^2$. When $k = 0$ the spectral power is equal to zero since $\lambda_0 = 0$. A constant potential does not produce any power in the network since the voltage between each pair of vertices is 0. This kind of power, proportional to the frequency (squared), is present in the Teager energy operator [16].

The Laplacian of a signal on a graph, with elements $\mathcal{L}_x(n)$, is a kind of generalized second order derivation on a graph, with graph Fourier transform coefficients $\lambda_k X(\lambda_k)$. Its inverse $\mathcal{L}_x^{-1}(n)$ with the graph Fourier transform coefficients $X(\lambda_k)/\lambda_k$, $\lambda_k \neq 0$, can be considered as a kind of generalized (double) integration on a graph as shown in Fig. 1 (bottom).

Example: Consider the signal and the graph in Fig. 1. The vertex-frequency distribution corresponding to the spectrogram is shown in Fig. 1. This vertex-frequency representation is not well concentrated. Its vertex-frequency concentration will be improved using the local distribution $P(n,k)$. This distribution, along with its marginal property values, is shown in Fig. 2. Marginal values are exact within the computer precision.

B. Vertex-Frequency Energy Distributions

In classical signal processing, the energy of a signal is defined as

$$E = \sum_{n=1}^{N} x^2(n) = \sum_{n=1}^{N} x(n) \sum_{k=1}^{N} X(\lambda_k) u_k(n), \qquad (12)$$

which can be alternatively written as

$$E = \sum_{n=1}^{N}\sum_{k=1}^{N} x(n)X(\lambda_k)u_k(n) = \sum_{n=1}^{N}\sum_{k=1}^{N} E(n,k), \qquad (13)$$

where the vertex-frequency energy distribution is

$$E(n,k) = x(n)X(\lambda_k)u_k(n) = \sum_{m=1}^{N} x(n)x(m)u_k(m)u_k(n). \qquad (14)$$

This vertex-frequency distribution corresponds to the Rihaczek distribution in the time-frequency analysis. Its marginal properties are defined as

$$\sum_{n=1}^{N} E(n,k) = |X(\lambda_k)|^2 \quad \text{and} \quad \sum_{k=1}^{N} E(n,k) = x^2(n). \qquad (15)$$

The summation over the vertex index n produces the squared spectra $|X(\lambda_k)|^2$, while the summation over the spectral index k produces the signal power $x^2(n)$. The distribution $E(n,k)$, along with marginal properties for the signal presented in Fig. 1, is illustrated in Fig. 3. Comparing this distribution with the localized spectral decomposition in Fig. 1, we can conclude that the localization of signal energy in the joint vertex-frequency domain is improved. In the definition of distribution $E(n,k)$ the localization window is not used.

C. Relation with the Vertex-Frequency Spectrogram

The standard localized spectrum in the vertex-frequency analysis is defined in [11] as

$$LS_x(n,\lambda_k) = \sum_{i=1}^{N} x(i)h_n(i)u_k(i), \qquad (16)$$

where the localized version of the window for the vertex n and frequency index k is defined as

$$h_n(i)u_k(i) = \sum_{j=1}^{N} \left[H(\lambda_j)u_j(n)u_j(i)\right]u_k(i). \qquad (17)$$

For example, we can use $H(\lambda_j) = C\exp(-\lambda_j\tau)$. The squared value of this transform, corresponding to the spectrogram, is

$$LS_x^2(n,\lambda_k) = \sum_{i=1}^{N}\sum_{m=1}^{N} x(i)h_n(i)x(m)h_n(m)u_k(i)u_k(m)$$
$$= \sum_{i=1}^{N}\sum_{m=1}^{N} r(i,m,k)h_n(i)h_n(m), \quad (18)$$

where $r(i,m,k) = x(i)x(m)u_k(i)u_k(m)$. The energy vertex-frequency distribution (14) is then

$$E(n,k) = \sum_{m=1}^{N} r(n,m,k). \quad (19)$$

The spectrogram defined by (18) is a smoothed version of $E(n,k)$. In order to avoid smoothing in (18) along index i the localization window should behave as $h_n(i) \sim \delta(i-n)$. To avoid weighting over index m the localization window should behave as $h_n(m) \sim 1$. These two requirements are contradictory, meaning that the energy concentration in the spectrogram cannot achieve energy concentration of $E(n,k)$.

D. Smoothed Vertex-Frequency Energy Distributions

The marginal properties may remain unchanged if the energy distribution $E(n,k)$ is convolved with a smoothing function $\Pi_n(i,k)$. Smoothing over the vertex index i should be done taking into account the neighborhood of vertex n. A smoothed vertex-frequency distribution is defined as

$$D(n,k) = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{m=1}^{N} x(i)x(m)u_j(m)u_j(i)\Pi_n(i,j-k), \quad (20)$$

where $\Pi_n(i,k)$ is a two-dimensional smoothing kernel in the distribution domain. The time marginal property remains the same as in $E(n,k)$ if this kernel satisfies the condition $\sum_{i=1}^{N}\Pi_n(i,k) = 1$ for all n and k.

The window localized spectral representation $LS_x(n,\lambda_k)$ can be related to $D(n,k)$ using $\Pi_n(i,k) = h_n(i)\delta(k)$ yielding

$$D(n,k) = LS_x(n,\lambda_k)X(\lambda_k). \quad (21)$$

Obviously, if $\Pi_n(i,k) = \delta(i-n)\delta(k)$ then $D(n,k) = E(n,k)$.

The vertex-frequency distribution with the Choi-Williams type of smoothing kernel is presented in Fig. 4. This kind of distribution preserves the time marginal property. Vertex domain smoothing is performed taking into account the vertex neighborhood ordering [15]. Smoothed vertex-frequency distributions would be less sensitive to disturbances and possible cross-terms. Smoothing can be performed, for example, only along the spectral index or only within the vertex neighborhood. In this case, only one marginal property can be preserved.

IV. Conclusion

Energy forms of vertex-frequency representations have been introduced. These distributions do not require localization windows which are a significant drawback of the linear vertex-frequency representations. This kind of analysis can lead to many other generalizations based on the classical time-frequency analysis. Our current research is directed toward the application of the proposed approach to EEG signals from [6] and [12]. The initial results indicate that we can obtain concentration improvements as in the simulated examples.

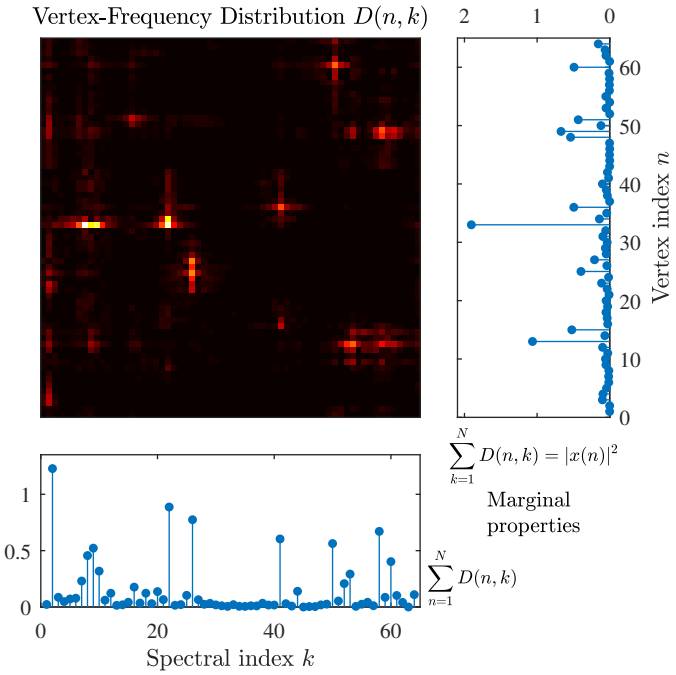

Fig. 4: A smoothed vertex-frequency distribution in the graph signal from Fig. 1.

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