Compressive Sensing of Signals Sparse in 2D Hermite Transform Domain

Miloš Brajović, Irena Orović, Miloš Daković, Srđan Stanković
Faculty of Electrical Engineering, University of Montenegro, 81000 Podgorica, Montenegro
milosb@ac.me

Abstract **– Compressive sensing (CS) of signals that exhibit sparsity in the domain of 2D Hermite transform (HT) is considered. The aim is to provide a successful reconstruction of randomly positioned missing samples. Gradient algorithm originally developed for the case of 1D HT as the domain of sparsity is applied as the reconstruction method, generalized and adapted for the case of signals sparse in 2D HT domain. This reconstruction approach minimizes the sparsity measure, with missing samples observed as minimization variables and updated in a steepest descent manner. Theoretical contributions are verified with numerical results.**

Keywords **– Compressive sensing; Hermite functions; Hermite transform; Sparse signal processing**

I. INTRODUCTION

Compressive sensing (CS) and sparse signal processing represent emerging research areas during the last decade [1]-[16]. Since the CS assumes the sparsity of signals in a transformation domain, these areas are closely connected [1]-[3]. The sparsity of a signal is defined as the number of non-zero transformation coefficients. Signals that appear in different applications have different sparsity domains. For example, digital images are sparse in the discrete cosine transform (DCT) domain. On the other side, QRS complexes, that are one of crucial segments of ECG signals, are sparse in the domain of discrete Hermite transform [10], [14], [16].

CS deals with signals with missing samples [1]-[15]. Namely, under the condition that a signal is sparse in a transformation domain, the aim is to recover all signal values from a reduced set of observations. The reduced set of measurements might be a consequence of a sampling strategy or simply their physical unavailability [1]-[9]. On the other side, in a number of applications signal values might be highly corrupted by noise. Thus, according to different robust techniques such as L-estimation, it is better to omit these highly corrupted signal values, and perform the analysis and processing of the signal with remaining values which are less corrupted. Hence, CS and sparse signal processing algorithms can be also applied in the reconstruction of the omitted highly corrupted signal values [4], [10]-[15]. Ether the samples are missing as a consequence of a sampling strategy (with the aim to reduce the data size) or they are intentionally omitted due to the corruption, the reconstruction problems can be treated as equivalent from the perspective of CS.

The basic idea behind the CS based reconstruction of missing samples is to find the solution of an undetermined system of equations that has the sparsest transform representation [1], [2], [10], [11]. The available observations, incorporated in equations of the undetermined system of equations, in fact define the conditions for the minimization of the sparsity measures. The sparsity can be directly measured using the ℓ_0-norm of the transform coefficients [1]-[4], [10]-[15]. This norm is equal to the number of non-zero transform coefficients. A reconstruction approach that can be considered as ℓ_0-norm based is presented in [11] and [12]. However, the since this norm is very sensitive to noise as well as to quantization effects, and moreover, it is not convex, minimization procedures cannot be directly applied, and other more robust norms are more commonly used in CS reconstruction algorithms. For instance, the sparsity is usually measured with ℓ_1-norm of the transform coefficients [1]-[16]. This relaxed reconstruction constraint has opened the possibility to apply a number of different reconstruction approaches. There are several reconstruction algorithms based on linear programing, for instance primal-dual interior point methods [1], [4], as well as other approaches - iterative gradient procedures such as Orthogonal Matching Pursuit (OMP), Gradient Pursuit, CoSaMP [6], [10], [13], [14]. An interesting steepest descent-based approach for the reconstruction of missing samples is presented in [13] and [14].

The Hermite transform (HT) and Hermite functions (HF) have attracted important research attention due to advantageous properties in certain applications, when compared with standard signal transforms, such as discrete Fourier transform (DFT) and DCT [10], [14]-[17] Representation and analysis, recognition and compression of QRS complexes [10], [14], [16], digital image processing [10], computed tomography, analysis of protein structure etc. [14] are just some representative applications of HT and HFs. HF and HT have very interesting properties regarding the possibilities of recursive calculations, that opened the way to fast calculation algorithms [16]. Signals that have a compact time-support usually have a concise HT due to the localization properties of HFs, with a small number of non-zero coefficients needed to represent such signals. These facts lead to applications of this transform in compression and recognition algorithms for ECG signals and their important parts – QRS complexes [14][16][17], as well as in image processing applications such as the digital image segmentation [17].

Recently, we have proposed a gradient-based reconstruction approach for signals with missing samples, that exhibit sparsity in the domain of 1D Hermite transform [14].

The ideas behind the approach are based on similar gradient-based reconstruction procedure presented in [13]. Since 2D HT represents the generalization of the 1D HT with very important applications in image processing [10], we have found the motivation to investigate the possibility of the generalization of the CS concepts to the 2D signal case. Namely, this paper deals with the reconstruction of missing samples in such signals. Herein, we formulate the CS problem and propose the generalization of the gradient reconstruction algorithm, originally developed for 1D signals sparse in HT domain to the 2D signal case. Presented theory is illustrated with numerical examples of successful missing samples reconstruction.

The rest of the paper is organized as follows. After Introduction, we revisit the 1D HT as well as the 2D HT in Section 2. The signals sparse in 2Dl HT transform are placed in the framework of CS in Section 3, where the generalized gradient-based algorithm is presented as well. Numerical results are given in Section 4, while the paper ends with concluding remarks.

II. Hermite Transform

A. One dimensional Hermite transform

One-dimensional continuous Hermite functions are defined via following recursion:

$$\psi_0(t,\sigma)=\frac{1}{\sqrt{\sigma\sqrt{\pi}}}e^{-\frac{t^2}{2\sigma^2}}, \qquad \psi_1(t,\sigma)=\sqrt{\frac{2}{\sigma\sqrt{\pi}}}\frac{t}{\sigma}e^{-\frac{t^2}{2\sigma^2}},$$
$$\psi_p(t,\sigma)=\frac{t}{\sigma}\sqrt{\frac{2}{p}}\,\psi_{p-1}(t,\sigma)-\sqrt{\frac{p-1}{p}}\,\psi_{p-2}(t,\sigma). \quad (1)$$

They are closely related with the p-th order Hermite polynomial:

$$H_p(t)=(-1)^p e^{t^2}\frac{d^p(e^{-t^2})}{dt^p} \quad (2)$$

as follows:

$$\psi_p(t,\sigma)=\left(\sigma 2^p n!\sqrt{\pi}\right)^{-1/2} e^{-t^2/2}H_p(t/\sigma). \quad (3)$$

The scaling factor σ controls the width of basis functions. For the sake of simplicity, in our analysis it will be assumed that $\sigma=1$ and will be omitted in further notation of HFs. The 1D Hermite expansion of the signal $f(t)$ has the following form:

$$f(t)=\sum_{p=0}^{N-1}c_p\psi_p(t) \quad (4)$$

where $\psi_p(t)$ is the p-th order Hermite basis function and N is the number of basis functions used in the expansion. In general, an infinite number of basis functions is needed for the representation of a continuous-time signal. Hermite coefficients are defined as:

$$c_p=\int_{-\infty}^{\infty}f(t)\psi_p(t)dt,\ p=0,1,\ldots,M-1. \quad (5)$$

Previous integral is calculated using the well-known Gauss-Hermite quadrature, which can be understood as the discrete HT of the observed signal:

$$c_p=\frac{1}{M}\sum_{n=1}^{M}\frac{\psi_p(t_n)}{[\psi_{M-1}(t_n)]^2}f(t_n),\ p=0,1,\ldots,M-1, \quad (6)$$

where the points t_n correspond to zeros of the M-th order Hermite polynomial (2). It is crucial to note that if the basis functions in expansion (4) are sampled at zeros of the M-th order Hermite polynomial, then the number of $N = M$ basis functions uniquely and completely represent a discrete signal $f(t_n)$ in the HT domain. In the other words, the expansion (4) for $N = M$ and with basis functions $\psi_p(t_m,\sigma)$, $p = 0, \ldots, M-1$ and (6) represent the Hermite transform pair.

B. Two-dimensional Hermite transform

Two-dimensional Hermite functions are defined as follows:

$$\psi_{pk}(x,y)=\frac{(-1)^p e^{x^2/2}}{\sqrt{2^p p!\sqrt{\pi}}}\frac{d^p(e^{-x^2})}{dx^p}\frac{(-1)^k e^{y^2/2}}{\sqrt{2^k k!\sqrt{\pi}}}\frac{d^k(e^{-y^2})}{dy^k} \quad (7)$$

where indexes p and k denote that it is the 2D function of order pk. Two 2D HFs are shown in Fig.1: $\psi_{12}(x_m,y_n)$ and $\psi_{52}(x_m,y_n)$. Obviously, according to (2) and (3), 2D HFs can be calculated as a product of corresponding 1D HFs:

$$\psi_{pk}(x,y)=\psi_p(x)\psi_k(y). \quad (8)$$

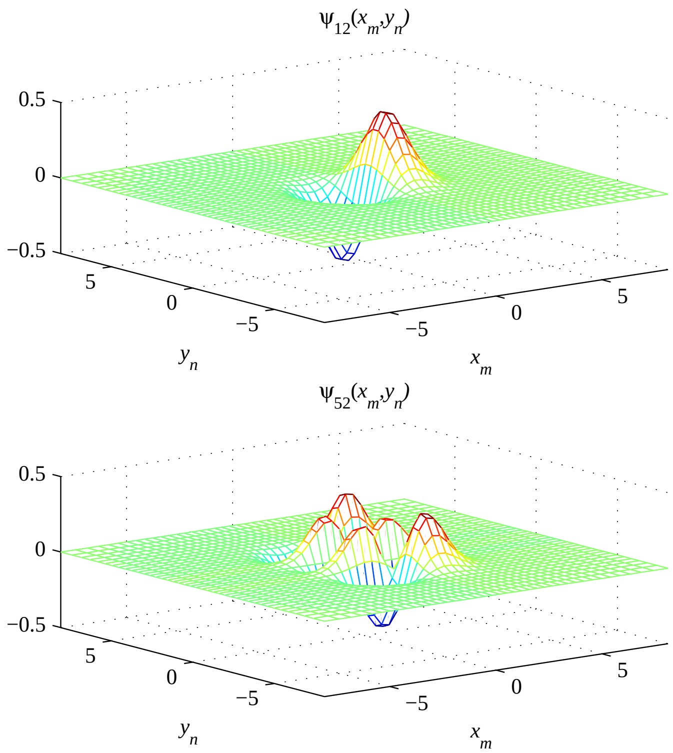

Figure 1. Two 2D Hermite functions

Let us assume that a 2D discrete signal $f(x_n, y_m)$ with dimensions M x N is sampled at roots of the M-th order Hermite polynomial in dimension x, and at roots of the N-th order Hermite polynomial in dimension y. Let us further assume that functions (7) i.e. (8) are also sampled at points x_n in the first dimension, and y_m in the second. Then, the 2D discrete Hermite expansion, i.e. the inverse discrete 2D HT reads:

$$f(x_m, y_n) = \sum_{p=0}^{M-1}\sum_{k=0}^{N-1} c_{pk}\psi_p(x_m)\psi_k(y_n) \quad (9)$$

with $m = 1,...,M$, $n = 1,...,N$. Having in mind (5) and (6) 2D Hermite coefficients are defined as:

$$c_{pk} = \frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{\psi_p(x_m)}{[\psi_{M-1}(x_m)]^2}\frac{\psi_k(y_n)}{[\psi_{N-1}(y_n)]^2} f(x_m, y_n), \quad (10)$$

with $p = 0,...,M-1$, $k = 0,...,N-1$. The Gauss-Hermite quadrature is used to calculate the integral of the form:

$$c_{pk} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\psi_p(x)\psi_k(y)dxdy, \quad (11)$$

which is defined for the case of continuous signal.

If we observe (9) and (10) it can be easily concluded that both direct and inverse HT can be calculated by using the 1D transform pair (4) and (6) over one variable with the other fixed, and then repeating the same calculation for the second variable.

III. CS IN THE DOMAIN OF 2D HERMITE TRANSFORM

Let us observe the signal of size M x N, sparse in the 2D HT domain:

$$f(x_m, y_n) = \sum_{i=1}^{K} A_i\psi_{p_i k_i}(x_m, y_n) \quad (12)$$

where A_i is used to denote the amplitude of the i-th signal component. Signal has non-zero coefficients at positions (p_i, k_i,) $i = 1,...,K$ in the 2D HT domain. The number of non-zero components K is known as sparsity. These coefficients represent signal support in the observed domain. The set consisted of all signal positions is denoted by $\mathbf{N}$.

Now assume that only $M_A N_A$ samples are available at positions $(x_m, y_n) \in \mathbf{M_A}$ (i.e. $(M - M_A)(N - N_A)$ samples are missing at random positions). If the signal satisfies that $K \ll MN$, according to CS theory, missing samples can be exactly reconstructed if certain conditions are met, as discussed in [1]-[4].

A. Gradient reconstruction algorithm

The basic idea behind the gradient-based algorithm presented in [14] is to set to zero the values in the signal at all missing samples positions, then to vary these values with a small, appropriately chosen step $\pm\Delta$ and then measure the 2D HT concentrations in both cases, in order to determine the gradient direction. The missing samples values are then updated in a steepest descent manner. A good starting value of the step can be obtained as:

$$\Delta = \max\left| f(x_m, y_n)\right|, \ (x_m, y_n) \in \mathbf{M_A}$$

Before the algorithm starts, the signal consisted of available signal samples and with zeros at missing samples positions is formed, according to :

$$y(x_m, y_n) = \begin{cases} f(x_m, y_n), & \text{for } (x_m, y_n) \in \mathbf{M_A} \\ 0, & \text{for } (x_m, y_n) \in \mathbf{N} \setminus \mathbf{M_A} \end{cases} \quad (13)$$

Then, for each iteration k the following steps are repeated, until the desired precision is obtained:

Step 1: For each missing sample, form two signals defined as:

$$y_1^{(k)}(x_m, y_n) = \begin{cases} y_1^{(k)}(x_m, y_n) + \Delta, & \text{for } (x_m, y_n) \in \mathbf{N} \setminus \mathbf{M_A} \\ y_1^{(k)}(x_m, y_n), & \text{for } (x_m, y_n) \in \mathbf{M_A} \end{cases},$$

$$y_2^{(k)}(x_m, y_n) = \begin{cases} y_2^{(k)}(x_m, y_n) - \Delta, & \text{for } (x_m, y_n) \in \mathbf{N} \setminus \mathbf{M_A} \\ y_2^{(k)}(x_m, y_n), & \text{for } (x_m, y_n) \in \mathbf{M_A} \end{cases},$$

Step 2: Calculate the finite difference of the signal transform measure

$$g(x_m, y_n) = \frac{1}{2\Delta}\left[\sum_p\sum_k \left|Y_{pk}^{+}\right| - \sum_p\sum_k \left|Y_{pk}^{-}\right|\right] \quad (14)$$

Note that Y_{pk}^{+} and Y_{pk}^{-} denote calculated 2D HTs of signals $y_1^{(k)}(x_m, y_n)$ and $y_2^{(k)}(x_m, y_n)$ respectively. Also note that $\sum_p\sum_k \left|Y_{pk}^{\pm}\right|$ represent ℓ_1 - norm of 2D HT [13], [14].

Step 3: Form the gradient matrix $\mathbf{G}^{(k)}$ of the same size as the analyzed signal $f(x_m, y_n)$ with elements defined as follows:

$$G^{(k)}(x_m, y_n) = \begin{cases} g(x_m, y_n), & \text{for } (x_m, y_n) \in \mathbf{N} \setminus \mathbf{M_A} \\ 0, & \text{for } (x_m, y_n) \in \mathbf{M_A} \end{cases} \quad (15)$$

where $g(x_m, x_n)$ is calculated in the Step 2.

Step 4: Correct the values of $y(n)$ using the gradient vector $\mathbf{G}^{(k)}$ with the steepest descent approach:

$$y^{(k+1)}(x_m, y_n) = y^{(k)}(x_m, y_n) - 2\Delta G^{(k)}(x_m, y_n). \quad (16)$$

High level of precision can be achieved with the decrease of Δ when the algorithm convergence slows down. This can be detected either by measuring reconstructed signal sparsity or by detecting oscillatory nature of the adjustments, [13] , [14].

IV. NUMERICAL RESULTS

Observe the signal of the form (12), with K = 4, positions $(p_1, k_1) = (1, 1)$, $(p_2, k_2) = (3, 2)$, $(p_3, k_3) = (5, 7)$, $(p_4, k_4) = (6, 9)$ with corresponding component amplitudes: $A_1 = 4.5, A_2 = 3$, $A_3 = -3$, and $A_4 = 0.2$. The signal size is 30×30.

In our example, 200 samples out of 900 are available (about 22%), while 700 samples are missing at random positions. The signal with missing samples is shown in Fig. 2 (first row). We have successfully applied the gradient algorithm presented in the previous Section. The reconstructed signal is shown in Fig. 2 (second row). The reconstruction MSE is below -120dB.

Moreover, we observe the 2D HT of the analyzed signal with missing samples, Fig. 3 (left). As the consequence of the missing samples in signal domain, large number of coefficients has significant non-zero values. The 2D HT of the reconstructed signal is shown on Fig 3 (right), with only 4 coefficients with significant values, which corresponds to the case when all samples available, proving that the reconstructed signal also has the original signal sparsity. The presented gradient algorithm successfully finds the values of missing samples that correspond to the sparsest possible solution of the observed CS problem.

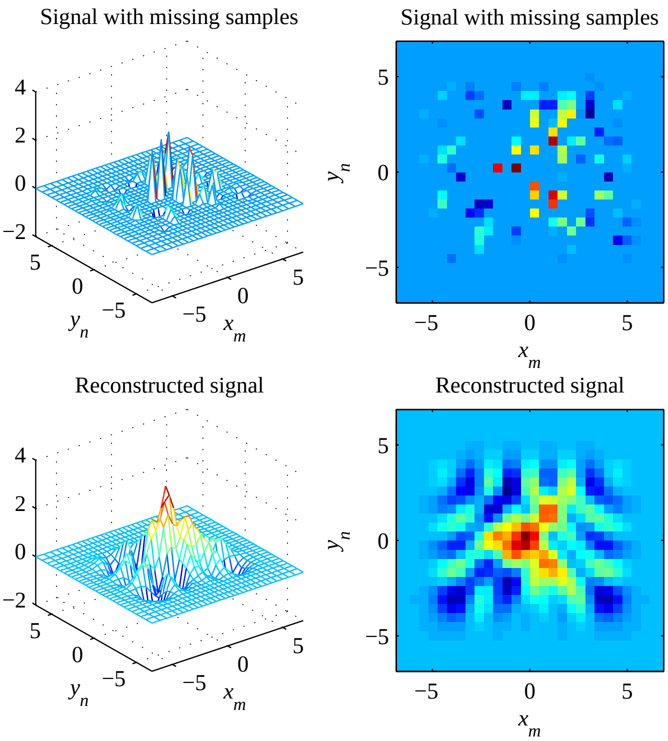

Figure 2. The reconstruction of missing samples: signal with missing samples (first row) and the reconstructed signal (second row)

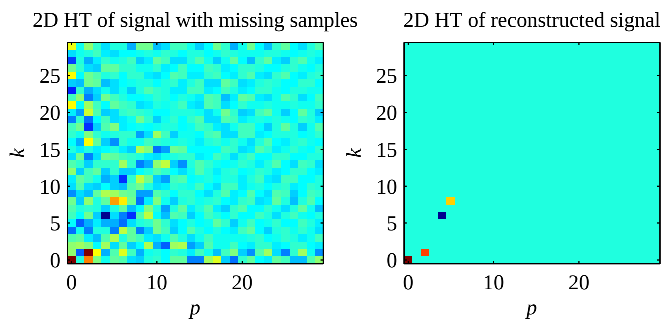

Figure 3. 2D HT coefficients of the signal with missing samples (left) and 2D HT coefficients of the reconstructed signal (right)

V. Conclusion

In this paper the reconstruction of randomly positioned missing samples of signals sparse in 2D HT domain is analyzed. Previously developed gradient algorithm is extended to 2D HT domain of sparsity. Numerical results confirm the presented theory. The successful reconstruction highly depends on signal sparsity and the number of missing samples. Successful reconstruction of sparse signals is guaranteed, if the well-known reconstruction conditions are met. Our current research includes the application of the presented algorithm to the parameterized 2D Hermite basis, with the aim to increase the sparsity of real signals in the analyzed domain.

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References

[1] D. Donoho, "Compressed sensing," *IEEE Trans. on Information Theory*, 52, (4), pp. 1289 – 1306, 2006.

[2] R. Baraniuk, "Compressive sensing," *IEEE Signal Processing Magazine*, 24, (4), pp. 118-121, 2007.

[3] M. Elad, *Sparse and Redundant Representations: From Theory to Applications in Signal and Image Processing*, Springer, 2010

[4] E. Candes, J. Romberg, and T.Tao, "Robust Uncertainty Principles: Exact Signal Reconstruction From Highly Incomplete Frequency Information," *IEEE Trans.on Inf. Theory*, 52(2), pp. 489-509, 2006.

[5] E. Sejdić, M. A. Rothfuss, M. L. Gimbel, M. H. Mickle, “Comparative Analysis of Compressive Sensing Approaches for Recovery of Missing Samples in an Implantable Wireless Doppler Device,” *IET Signal Processing*, vol. 8, no. 3, pp. 230-238, May 2014.

[6] M. Davenport, M. Duarte, Y. Eldar, G. Kutyniok, "Introduction to compressed sensing," Chapter in *Compressed Sensing: Theory and Applications*, Cambridge University Press, 2012.

[7] I. Candel, C. Ioana, B. Reeb, “Robust sparse representation for adaptive sensing of turbulent phenomena,” *IET Signal Processing*, vol. 8, no. 3, 2014, pp. 285-290

[8] X. Li, G. Bi, “Image reconstruction based on the improved compressive sensing algorithm,” *IEEE Int. Conference on Digital Signal Processing (DSP)*, pp. 357 – 360, Singapore 2015

[9] H. Rauhut, R. Ward, “Sparse Legendre expansions via l1-minimization,” *J. Approx. Theory*, 164(5):517-533, 2012.

[10] S. Stanković, I. Orović, and E. Sejdić, *Multimedia signals and systems*, Springer - Verlag, 2012

[11] S. Stanković, I. Orović, and LJ. Stanković, "An Automated Signal Reconstruction Method based on Analysis of Compressive Sensed Signals in Noisy Environment, "*Sig. Proc.*, vol. 104, pp. 43 - 50, 2014.

[12] S. Stanković, and I. Orović, “An Approach to 2D Signals Recovering in Compressive Sensing Context,” *submitted to Circuits, Systems & Signal Processing*, https://arxiv.org/ftp/arxiv/papers/1502/1502.05980.pdf

[13] LJ. Stanković, M. Daković, and S. Vujović, “Adaptive Variable Step Algorithm for Missing Samples Recovery in Sparse Signals,” *IET Signal Processing*, vol. 8, no. 3, pp. 246 -256, 2014.

[14] M. Brajović, I. Orović, M. Daković, and S. Stanković, "Gradient-based signal reconstruction algorithm in the Hermite transform domain," *Electronics Letters, Volume 52, Issue 1, pp.41-43, 2016*

[15] M. Brajović, I. Orović, M. Daković, and S. Stanković, “The Analysis of Missing Samples in Signals Sparse in the Hermite Transform Domain,” *23rd Telecommunications Forum TELFOR*, Belgrade, 2015

[16] A. Sandryhaila, S. Saba, M. Puschel, J. Kovacevic, "Efficient compression of QRS complexes using Hermite expansion, " *IEEE Transactions on Signal Proccessing*, vol.60, no.2, pp.947-955, 2012.

[17] J.-B. Martens, "The Hermite transform—Applications, " *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 38, no. 9, pp. 1607–1618, 1990.