

# REKONSTRUKCIJA NESTACIONARNIH SIGNALA SA NEDOSTAJUĆIM ODBIRCIMA PRIMJENOM S-METODA I GRADIJENTNOG ALGORITMA ZA REKONSTRUKCIJU RECONSTRUCTION OF NON-STATIONARY SIGNALS WITH MISSING SAMPLES USING S-METHOD AND A GRADIENT BASED RECONSTRUCTION ALGORITHM

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**Sadržaj:** *U radu je razmatrana problematika rekonstrukcije nestacionarnih signala sa nedostajućim odbircima korišćenjem mjera koncentracije vremensko-frekvencijskih reprezentacija i gradijentnog algoritma za rekonstrukciju. Kao vremensko-frekvencijska reprezentacija korišćen je S-metod. Teorijska razmatranja potkrijepljena su odgovarajućim primjerima, identifikovani su postojeći problem, kao i ideje za budući rad.*

**Abstract:** *This paper addresses the reconstruction problem of non-stationary signals with missing samples, using concentration measures of time-frequency representations and a gradient-based reconstruction algorithm. As an example of a time-frequency representation, S-method is used in the proposed approach. Theoretical considerations are illustrated through the several examples, and the noticeable problems and the further work ideas are identified.*

## 1. INTRODUCTION

Compressive sensing (CS) and sparse signals analysis have drawn significant research attention in the last decade [1]-[11]. These areas are closely related since the CS can be applied with the signal sparsity assumption in a transformation domain. Time-frequency signal analysis can be related with CS and sparse signals analysis in several aspects. It is recently proposed that S-method can be used not only for focusing the ISAR radar images and in that way improving the radar signal sparsity, but also it can be used as a sparse transform directly in the recovery process, in cases when these signals have missing samples (e.g. after noise reduction) [10]. The L-statistics, stationary and non-stationary signal separation and reconstruction are other representative illustrations of these fundamental connections between these areas [9]-[11]. The last few decades of the intensive research in the area of time-frequency produced a large number of algorithms, methods and different mathematical approaches for the analysis and processing of non-stationary signals, with a large number of applications in several areas [12]-[15].

The area of CS deals with signals which have a certain number of missing samples. The reduced set of observations in CS is usually a consequence of a strategically chosen sampling method. Usually, signal samples can be intentionally omitted due to high noise corruption, or eliminated using robust techniques and L-statistics. All these cases can be treated as equivalent problems in the context of CS [10]. Mathematical foundation of the CS usually lies in fact that it is possible to reconstruct a sparse signal by interpreting the problem as an undetermined set of linear equations using optimization approaches such as linear programming [1], [3], [4]. Other CS approaches and algorithms include signal processing techniques adopted based on the analysis on missing samples consequences, for example, on the signal transformation domain [7]. If the CS problem can be interpreted as an undetermined set of linear

equations, then it can be solved using a direct search approach. However, this approach is identified as NP hard problem, very time-demanding and not possible to solve in real scenarios. The problem can be converted to a linear programming form, and solved by, for example, using the primal-dual algorithm, as it is commented in [2]. Other approaches for the reconstruction include gradient-based methods, such as ones introduced in [2], [5] and [10], also recognized as suitable in the context of our problem. These methods are iterative procedures based on a gradient descent optimization.

The sparsity is a fundamental condition needed for the successful CS, and the performances and outcomes of the reconstruction process highly depend on the suitable choice of the transform domain in which the signal is sparse. The basic idea behind our analysis lies in fact that time-frequency representations concentrate the signal energy around the instantaneous frequency of the signal [12], [13]. Better concentration implies a smaller number of non-zero values of the time-frequency representation, and thus, it can be interpreted as a sparsity measure. Concentration measures of time-frequency representations have been studied in [15], and put into the context of CS in [2] and [10]. Concentration measures of linear transforms are used in gradient-based algorithms as a measure of signal sparsity in the transformation domain [2], [5].

Within this paper, we will try investigate the possibility to relate the concept of sparsity with high level of concentration of some time-frequency representations (TFRs) via concentration measures, with aim to reconstruct non-stationary signals with missing samples. Although a time-frequency representation can be considered as sparse for a specific class of signal, and used for its reconstruction, such as in [10] and [11], our main goal is to investigate the concept of sparsity through concentration measures for the general class of non-stationary signals. As an example of highly concentrated representation, S-method which is in detail

studied in [13] and [14] is used in this paper in a combination with the complex gradient reconstruction algorithm introduced in [5]. The main reason for using the gradient-based algorithm instead of conventional compressive sensing algorithms lays in fact that S-method (as well as most of highly concentrated representations) as a domain of sparsity (in the context of concentration measures) has a non-linear relation with the signal, as it is stressed in [10].

## 2. BASIC THEORY AND PROBLEM DEFINITION

The discrete S-method, as an example of a highly concentrated time-frequency representation is defined with:

$$SM_{x,L}(n,k) = |STFT(n,k)|^2 + 2\text{Re}\left\{\sum_{i=1}^L STFT(n,k+i)STFT^*(n,k-i)\right\}, \quad (1)$$

where the parameter  $L$  with typical values between 3 and 5 defines the quality (concentration) improvement of the spectrogram  $|STFT(n,k)|^2$  towards the Wigner distribution, as explained in detail in [13] and [14]. A suitably chosen value of  $L$  suppresses the rise of undesired components known as cross terms.  $STFT(n,k)$  denotes the Short-time Fourier transform (STFT), which is for a discrete signal  $x(n)$  defined with:

$$STFT(n,k) = \sum_{m=-N_w/2}^{N_w/2-1} x(n+m)w(m)e^{-j\frac{2\pi}{N_w}km}, \quad (2)$$

with  $w(m)$  being the window function of length  $N_w$ .

The concentration measures were studied and widely used for the optimization of time-frequency representations [13], [15]. Concentration of a TFR  $\rho(n,k)$  of the signal  $x(n)$  can be defined with:

$$\Upsilon_\rho^p = \sum_n \sum_k |\rho(n,k)|^{1/p}. \quad (3)$$

Different values of  $p$  define different concentration measures. Highly concentrated TFRs have smaller number of non-zero values in the time-frequency plane, and thus, lower values of concentration measures. As it is known, in the context of CS the most suitable measure of sparsity, i.e. the concentration measure for counting the number of non-zero values is the  $l_0$  norm, obtained when  $p \rightarrow \infty$  in (3):

$$\Upsilon_{SM}^0 = \|SM_{x,L}\|_0 = \sum_n \sum_k |SM_{x,L}(n,k)|^0 \quad (4)$$

However, the minimization of this norm would imply a combinatorial search, which is a known NP hard problem. Additionally, since it only counts non-zero elements, even the smallest disturbances may cause problems in the minimization process [1]. This is the reason why in the CS other norms, such as  $l_1$ , are used.

According to the definitions of the S-method and the STFT (1) and (2), it can be easily concluded that the  $l_1$  equivalent norm (concentration measure) can be obtained by setting  $p = 2$  in (3) as:

$$\Upsilon_{SM}^{1/2} = \|SM_{x,L}(n,k)\|_{1/2} = \sum_n \sum_k |SM_{x,L}(n,k)|^{1/2}. \quad (5)$$

If the signal  $x(n)$  with  $M_m$  missing samples is considered, the aim of reconstruction is to find a solution (i.e.

the values of missing samples) which gives the sparsest S-method of signal. If we denote with  $\mathbf{N}_x$  the set consisted of positions of available samples, the problem of reconstruction can be formulated as:

$$\min \sum_n \sum_k |SM_{y,L}(n,k)|^{1/2} \quad (6)$$

subject to  $y(n) = x(n)$  for  $n \in \mathbf{N}_x$ .

where  $y(n)$  denotes the reconstructed signal.

## 3. SOLVING THE MINIMIZATION PROBLEM USING THE GRADIENT BASED ALGORITHM

As discussed in the Introduction, previous optimization problem can be successfully solved with a gradient approach. The basic idea behind the gradient based algorithm presented in [2], [5] and [10] is to set to zero the values in the signal at all missing samples positions, and then to vary these values with a small, appropriately chosen step  $\pm\Delta$ . Since we aim to reconstruct a complex signal, both real and imaginary parts of missing samples should be varied with the step  $\Delta$ . A good starting value of the step can be obtained as  $\Delta = \max|x(n)|, n \in \mathbf{N}_x$ .

Before the procedure starts, the signal  $y(n)$  with zeros at the positions of missing samples is formed:

$$y(n) = \begin{cases} x(n), & \text{for } n \neq n_i \\ 0, & \text{for } n = n_i \end{cases} \quad (7)$$

Then, for each iteration  $k$  the following steps are repeated, until the desired precision is obtained:

*Step 1:* For each missing sample at the position  $n_i$ , form four signals defined as:

$$y_1^{(k)}(n) = \begin{cases} y^{(k)}(n) + \Delta, & \text{for } n = n_i \\ y^{(k)}(n), & \text{for } n \neq n_i \end{cases},$$

$$y_2^{(k)}(n) = \begin{cases} y^{(k)}(n) - \Delta, & \text{for } n = n_i \\ y^{(k)}(n), & \text{for } n \neq n_i \end{cases},$$

$$y_3^{(k)}(n) = \begin{cases} y^{(k)}(n) + j\Delta, & \text{for } n = n_i \\ y^{(k)}(n), & \text{for } n \neq n_i \end{cases},$$

$$y_4^{(k)}(n) = \begin{cases} y^{(k)}(n) - j\Delta, & \text{for } n = n_i \\ y^{(k)}(n), & \text{for } n \neq n_i \end{cases}.$$

*Step 2:* Estimate the real and imaginary gradient parts as differences of the concentration measures:

$$g_r(n_i) = \sum_n \sum_k |SM_{y_1,L}(n,k)|^{1/2} - \sum_n \sum_k |SM_{y_2,L}(n,k)|^{1/2}$$

$$g_i(n_i) = \sum_n \sum_k |SM_{y_3,L}(n,k)|^{1/2} - \sum_n \sum_k |SM_{y_4,L}(n,k)|^{1/2}$$

*Step 3:* Form the gradient vector  $\mathbf{G}^{(k)}$  of the same length as the analyzed signal  $x(n)$  with elements, defined as follows:

$$\mathbf{G}^{(k)}(n) = \begin{cases} g_r(n_i) + jg_i(n_i), & \text{for } n = n_i \\ 0, & \text{for } n \neq n_i \end{cases} \quad (8)$$

where  $g_r(n_i)$  and  $g_i(n_i)$  are calculated in the Step 2.

*Step 4:* Correct the values of  $y(n)$  using the gradient vector  $\mathbf{G}^{(k)}$  with the steepest descent approach:

$$y^{(k+1)}(n) = y^{(k)}(n) - \frac{1}{NN_w} \mathbf{G}^{(k)}(n). \quad (9)$$

As (9) is proportional to the error  $y(n) - x(n)$  for both real and imaginary parts, the missing values will converge to the true signal values. In order to obtain a high level of precision, the step  $\Delta$  should be decreased when the algorithm convergence slows down. In this paper the fixed step is used.

Since the values of missing samples are varied, the measure gradient enables approaching the optimal point which minimizes the measure of the S-method, meaning that the solution which gives the smallest number of non-zero values of the S-method is obtained. Until the optimal point is approached, the zero-valued or inaccurate missing samples cause a larger number of non-zero values, as it is analyzed in [7], and thus, the larger concentration measures. The satisfactory reconstruction results are obtained as long as the S-method concentration is high.

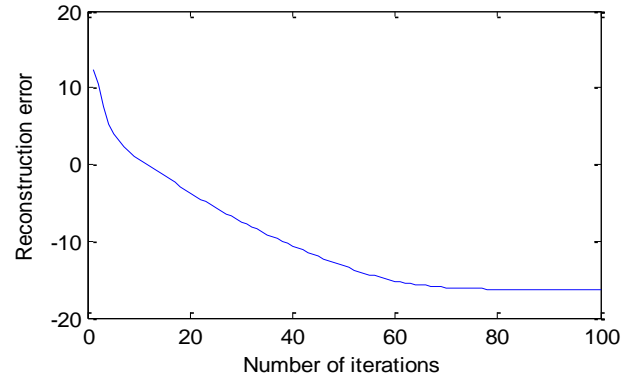


Figure 3: The reconstruction error during the iterations in the case of mono-component LFM signal

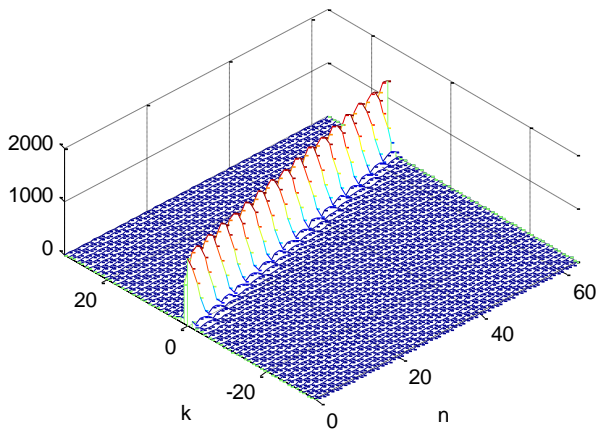


Figure 1: S-method of the LFM signal from the first example

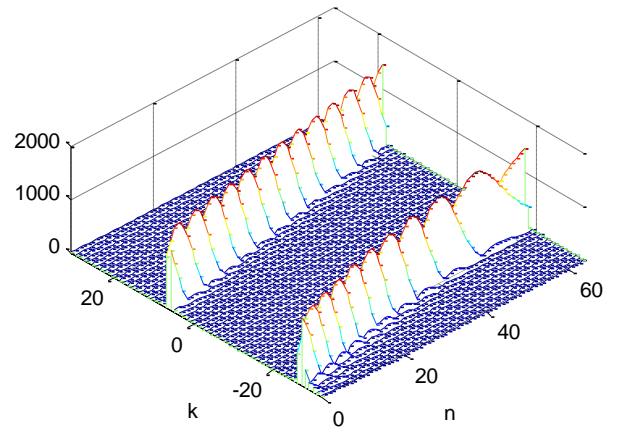


Figure 4: S-method of the two-component signal analyzed in the second example.

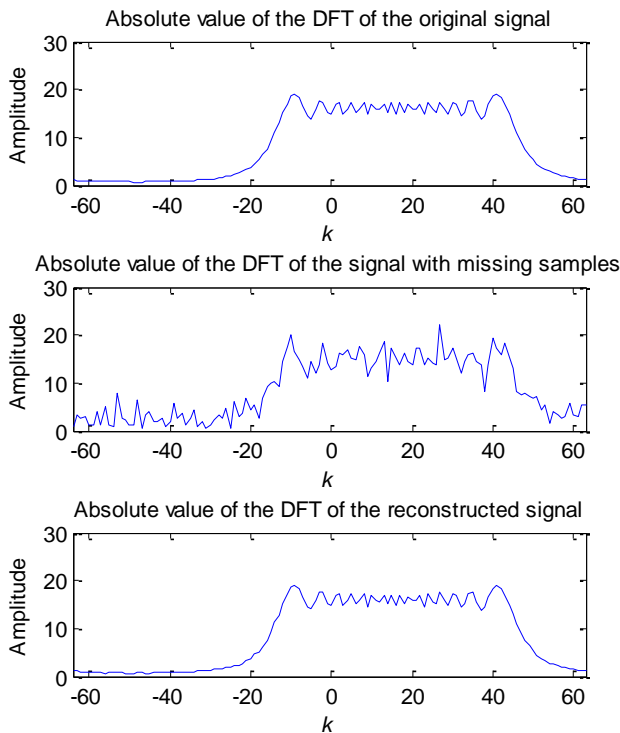


Figure 2: Reconstruction of LFM signal from the first example using S-method and gradient algorithm.

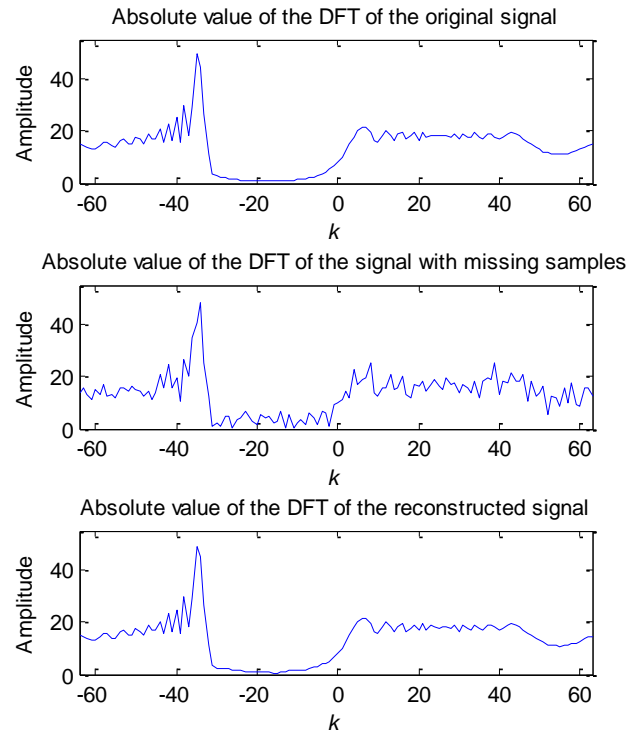


Figure 5: Reconstruction of the multicomponent signal from the second example using S-method and gradient algorithm.

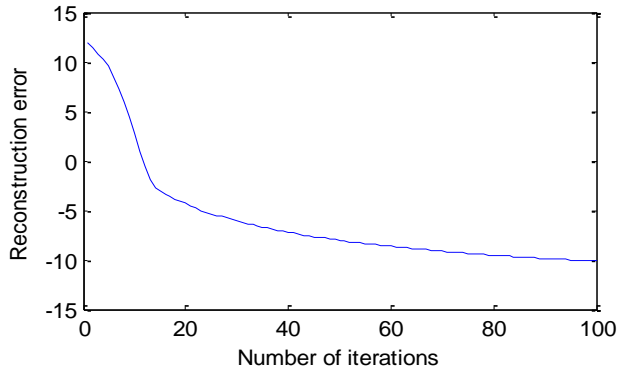


Figure 6: The reconstruction error during the iterations in the case of multicomponent signal

#### 4. EXAMPLES AND COMMENTS

*Example 1:* In order to illustrate the described reconstruction procedure, a LFM signal of length  $N = 128$ , with 15 missing samples is considered:

$$x(n) = \exp\left(-j200(n/128)^2 - j100(n/128)\right)$$

S-method with  $L = 6$  and Hanning window of length  $N_w = 64$  is used. S-method of the signal  $x(n)$  without missing samples, calculated with these parameters is shown on Fig. 1. The absolute value of the DFT of the considered signal with all samples, considered signal with missing samples and of the reconstructed signal are shown on Fig. 2. DFT is used to show the reconstruction results in order to emphasize the differences between signals. The MSE defined as:

$$MSE = 10 \log_{10} \left( |y(n) - x(n)|^2 \right) \quad (10)$$

is shown on Fig. 3 to track the algorithm convergence.

*Example 2:* The problem of the reconstruction of a two-component signal:

$$x(n) = \exp\left(-j40\pi(n/128)^3 + j100\pi(n/128)^2 + j30n\right) + \exp\left(j50\pi(n/128)^2\right)$$

of length  $N = 128$  is considered, having 10 missing samples. The S-method is calculated with  $L = 3$  and with Hanning window of length  $N_w = 64$ . The S-method of the analyzed signal with all samples available is shown on Fig. 4. Reconstruction results are shown on Fig 5, while the error (10) is shown on Fig. 6. In this example a signal consisted of a linear and quadratic FM component is considered.

This example emphasizes the fact that a suitable choice of the parameters  $L$  and  $N_w$  is significant for the success of the reconstruction, since the S-method concentration depend on these values. It is crucial to set the S-method parameters such that the highest possible concentration is achieved, in order to obtain a successful reconstruction. If the signal non-stationarity is such that the sparsity property cannot be satisfied in the analyzed domain, the more concentrated time-frequency representations should be considered. The problem of the optimal time-frequency representation choice in the sense of concentration is analyzed in detail in [12] and [13]. The satisfactory results were obtained for two-component signals consisted of linear and quadratic FM components.

#### 5. CONCLUSION AND FURTHER RESEARCH

Reconstruction of non-stationary signals using an iterative gradient based algorithm and S-method is addressed. The theoretical considerations are illustrated with two examples. The determination of the maximal possible number of signal samples which can guarantee the reconstruction will be a part of our further research, as well as the reconstruction using other highly concentrated representations. Our further work will also address the possibility to obtain a higher level of sparsity in the time-frequency domain for the general class of non-stationary signals with arbitrary non-stationarity level.

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