Gradient-based signal reconstruction algorithm in the Hermite transform domain

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Abstract: An algorithm for compressive sensing reconstruction of signals in the Hermite expansion domain is proposed. The compressive sensing problem is formulated in the Hermite framework, allowing fast and efficient reconstruction of missing data by exploiting the concentration of signal's representation in the Hermite basis.

Introduction: The Hermite expansion approach has drawn significant attention in certain signal processing applications where classical tools, including also the Fourier transform, are not suitable for analysis. Namely, the Hermite expansion is an orthogonal transform used in image processing, tomography, analysis of protein structure, biomedicine [1-3]. The Hermite functions have been used as a suitable basis for representation and compression of QRS complexes of ECG signals, important for diagnosis and medical treatment. Particularly, the application in compression algorithms shows that the reconstruction of ECG signals can be done using a few Hermite coefficients [1].

Compressive sensing (CS) as an alternative sampling theory assumes signal sparsity in a certain transform domain in order to achieve successful reconstruction of missing data. A reduced set of observations in CS may appear as a consequence of a sampling strategy, or by omitting samples highly corrupted by noise. The reconstructed signal can be obtained by using the ℓ_1-norm minimization via convex optimization algorithms [4-6], which could be complex in terms of the realization and the number of iterations. Here, we provide an iterative reconstruction approach based on a steepest descent method using ℓ_1 norm minimization in the Hermite transform domain. Namely, the CS framework is defined in the context of the Hermite expansion, while the achieved results demonstrate successful reconstruction using the gradient-based solution [7]. The proposed approach provides faster performance compared to the other convex algorithms such as the commonly used primal-dual method within the ℓ_1-magic toolbox [8].

Hermite expansion: The p-th order Hermite function can be related with the p-th order Hermite polynomial:

$$\psi_p(t,\sigma)=\left(\sigma 2^n n!\sqrt{\pi}\right)^{-1/2} e^{-t^2/2} H_p(t/\sigma)\ , \tag{1}$$

where the parameter σ can be used to stretch or compress Hermite functions, in order to match the analysed signal [1]. A signal representation using the Hermite basis is referenced as Hermite expansion [1-3]:

$$f(t)=\sum_{p=0}^{\infty} c_p \psi_p(t,\sigma)\ . \tag{2}$$

For a continuous signal $f(t)$, an infinite number $N\to\infty$ of Hermite functions is needed for an accurate expansion. However, in practice, a finite number of N Hermite functions is used, as an approximation of the signal. The p-th order Hermite coefficient c_p is defined as:

$$c_p=\int_{-\infty}^{\infty} f(t)\psi_p(t)dt\ . \tag{3}$$

. If Hermite functions are sampled at zeros of the N-th order Hermite polynomial, then the summation (2) becomes a finite orthonormal representation for the case of discrete signals, [1]. In numerical calculation, the quadrature approximations (as a discrete form of the Hermite expansion) are used to obtain integral in (3). For instance, the Gauss-Hermite quadrature is defined as:

$$c_p \approx \frac{1}{N}\sum_{n=1}^{N}\frac{\psi_p(x_n)}{\left[\psi_{N-1}(x_n)\right]^2} f(x_n)\ , \tag{4}$$

where x_n denotes zero of the N-th Hermite polynomial. For a signal of length N, the complete set of discrete Hermite functions consists of exactly N functions [1]. In some applications, a smaller number of Hermite functions (compared to signal length) can be used [3]. To simplify the notation, in the sequel the argument x_n will be replaced with the order n (of the Hermite polynomial zeros). The expansion using N Hermite functions can be written in matrix form. First, we define the Hermite transform matrix $\mathbf{W_H}$ (of size $N\times N$):

$$\mathbf{W_H}=\frac{1}{N}\begin{bmatrix} \frac{\psi_0(1)}{(\psi_{N-1}(1))^2} & \frac{\psi_0(2)}{(\psi_{N-1}(2))^2} & \cdots & \frac{\psi_0(N)}{(\psi_{N-1}(N))^2} \\ \frac{\psi_1(1)}{(\psi_{N-1}(1))^2} & \frac{\psi_1(2)}{(\psi_{N-1}(2))^2} & \cdots & \frac{\psi_1(N)}{(\psi_{N-1}(N))^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\psi_{N-1}(1)}{(\psi_{N-1}(1))^2} & \frac{\psi_{N-1}(2)}{(\psi_{N-1}(2))^2} & \cdots & \frac{\psi_{N-1}(N)}{(\psi_{N-1}(N))^2} \end{bmatrix} \tag{5}$$

If the vector of Hermite coefficients is: $\mathbf{c}=\left[c_0,c_1,...,c_{N-1}\right]^T$, and vector of M signal samples is: $\mathbf{f}=[f(1),f(2),...,f(N)]^T$, then we have:

$$\mathbf{c}=\mathbf{W}_H\mathbf{f}\ . \tag{6}$$

Having in mind the Gauss-Hermite approximation (4), the inverse matrix $\mathbf{W_H^{-1}}$ contains N Hermite functions is given by:

$$\mathbf{\Psi}=\begin{bmatrix} \psi_0(1) & \psi_0(2) & \dots & \psi_0(N) \\ \psi_1(1) & \psi_1(2) & \cdots & \psi_1(N) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{N-1}(1) & \psi_{N-1}(2) & \cdots & \psi_{N-1}(N) \end{bmatrix}=\mathbf{W}_H^{-1}\ . \tag{7}$$

Now, the Hermite expansion can be defined as follows:

$$\mathbf{f}=\mathbf{W}_H^{-1}\mathbf{c}=\mathbf{\Psi c}\ . \tag{8}$$

Compressive sensing problem formulation: Let assume that the compressive sensing is done using a random selection of M_A signal values modelled by a random measurement matrix $\mathbf{\Phi}$:

$$\mathbf{y_{cs}}=\mathbf{\Phi f}=\mathbf{\Phi\Psi c}=\mathbf{A_{cs}c}\ . \tag{9}$$

Here, $\mathbf{y_{cs}}$ denotes the vector of available samples, matrix $\mathbf{A_{cs}}$ is obtained from the inverse Hermite transform matrix $\mathbf{\Psi}$ by omitting the rows corresponding to the missing samples. Hence, we deal with undetermined system of M_A linear equations and N unknowns. Although this system may have infinitely many solutions, the idea is to search for the sparsest one. Thus, the reconstruction problem can be defined as:

$$\min\|\mathbf{c}\|_{\ell_1}\ \textit{subject to}\ \mathbf{y}_{cs}=\mathbf{A}_{cs}\mathbf{c}\ . \tag{10}$$

Gradient based reconstruction algorithm: A previous minimization problem can be solved by adapting the use of gradient descent [7] in the Hermite transform domain. The idea of the proposed method is to iteratively recover the values of missing samples, by a small appropriately estimated gradient-based step. The ℓ_1 norm behaviour is examined in the Hermite expansion domain acting as a sparsity measure.

Assume that the positions of available samples are defined by the set $\mathbf{\Theta}$, where $\mathbf{\Theta}$ consists of M_A<<N elements and $\mathbf{\Theta}\subset\mathbf{N}$={1, 2, .., N}. Denote by n_i indices defined by:

$$n_i=\begin{cases} i, & i\in\mathbf{\Theta} \\ 0, & i\notin\mathbf{\Theta} \end{cases}.$$

The missing samples positions are form the set $\bar{\mathbf{\Theta}}$, such that $\mathbf{\Theta}\cup\bar{\mathbf{\Theta}}=\mathbf{N}$. The assumed signal sparsity is $K<<N$.
The algorithm starts from the column vector $\mathbf{y}$ which contains available samples $\mathbf{y_{cs}}$ and zero values at the positions of missing samples. Hence, $\mathbf{y}$ can be defined as: $y(n)=f(n)$, for $n\in\mathbf{\Theta}$, otherwise $y(n)$=0 (with $f(n)$ being the original signal samples). Assume that the initial value of the step is $\Delta=\max\left(|\mathbf{y}|\right)$. Each iteration (denoted by k) consists of the following steps.

1) Form a matrix $\mathbf{Y}$ with N repeated vectors $\mathbf{y}$:

$$\mathbf{Y}^{(k)}=\mathbf{y}^{(k)}\mathbf{1}_{1\times N}=\left[\mathbf{y}^{(k)}\quad \mathbf{y}^{(k)}\quad \dots\quad \mathbf{y}^{(k)}\right],$$

where the notation $\mathbf{1_{1xN}}$ is used for all-ones matrix of dimensions 1xN.

2) Calculate two test matrices a follows:

$$\mathbf{Y}_{\Delta+}^{(k)}=\left[\mathbf{y}_{1\Delta+}^{(k)}\quad \mathbf{y}_{2\Delta+}^{(k)}\quad \cdots\quad \mathbf{y}_{N\Delta+}^{(k)}\right]=\mathbf{Y}^{(k)}+\mathbf{\Delta} \tag{11}$$

$$\mathbf{Y}_{\Delta-}^{(k)}=\left[\mathbf{y}_{1\Delta-}^{(k)}\quad \mathbf{y}_{2\Delta-}^{(k)}\quad \cdots\quad \mathbf{y}_{N\Delta-}^{(k)}\right]=\mathbf{Y}^{(k)}-\mathbf{\Delta} \tag{12}$$

For better understanding, the previous matrices can be written in an expanded form given by:

$$\mathbf{Y}_{\Delta\pm}^{(k)} = \mathbf{Y}^{(k)} \pm \boldsymbol{\Delta} = \mathbf{y}^{(k)}\mathbf{1}_{1\times N} \pm \boldsymbol{\Delta} =$$
$$= \begin{bmatrix} y^{(k)}(1) & & y^{(k)}(1) \\ y^{(k)}(2) & \cdots & y^{(k)}(2) \\ \vdots & & \vdots \\ y^{(k)}(N) & & y^{(k)}(N) \end{bmatrix} \pm \Delta \begin{bmatrix} \delta(1-n_1) & 0 & \cdots & 0 \\ 0 & \delta(2-n_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta(N-n_N) \end{bmatrix}_{N\times N} .$$

where $\boldsymbol{\Delta}$ is obtained by multiplying constant Δ by a diagonal matrix with elements $\delta(n-n_i)$, for n, i={1, 2, … , N}.

3) Based on calculated test matrices $\mathbf{Y}_{\Delta+}^{(k)}$ and $\mathbf{Y}_{\Delta-}^{(k)}$, the gradient vector **G** is calculated as:

$$\mathbf{G}^{(k)} = \frac{1}{2\Delta}\left[\left\|\mathcal{H}^{(c)}\{\mathbf{Y}_{\Delta+}^{(k)}\}\right\|_{\ell_1}^{(c)} - \left\|\mathcal{H}^{(c)}\{\mathbf{Y}_{\Delta-}^{(k)}\}\right\|_{\ell_1}^{(c)}\right] =$$
$$= \frac{1}{2\Delta}\left[\left[\left\|\mathbf{W}_H\mathbf{y}_{1\Delta+}^{(k)}\right\|_{\ell_1} \;\; \left\|\mathbf{W}_H\mathbf{y}_{1\Delta+}^{(k)}\right\|_{\ell_1} \;\; \cdots \;\; \left\|\mathbf{W}_H\mathbf{y}_{1\Delta+}^{(k)}\right\|_{\ell_1}\right] - \right. \tag{13}$$
$$\left. - \left[\left\|\mathbf{W}_H\mathbf{y}_{1\Delta-}^{(k)}\right\|_{\ell_1} \;\; \left\|\mathbf{W}_H\mathbf{y}_{1\Delta-}^{(k)}\right\|_{\ell_1} \;\; \cdots \;\; \left\|\mathbf{W}_H\mathbf{y}_{1\Delta-}^{(k)}\right\|_{\ell_1}\right]\right]$$

with $\mathcal{H}^{(c)}\{\cdot\}$ being the operator that calculates the Hermite coefficients along the matrix columns, and operator $\|\cdot\|_{\ell_1}^{(c)}$ calculates ℓ_1 norm for each column separately. Note that gradient vector has zero values for signal samples at available samples positions.

4) Finally, the signal vector **y** is adjusted as follows:

$$\mathbf{y}^{(k+1)} = \mathbf{y}^{(k)} + \mathbf{G}^{(k)} / (2\Delta) \tag{14}$$

As $\mathbf{G}$ is proportional to the error $\mathbf{y}-\mathbf{f}$, the missing values will converge to the true signal values. In order to obtain a high reconstruction precision, the step Δ should be reduced when adjustments in (15) does not improve precision. This can be detected either by measuring reconstructed signal sparsity or by detecting oscillatory nature of the adjustments, [7].

Experimental evaluation: Let us observe the signal in the form:

$$y_{CS}(n) = \sum_{i=1}^{K} A_i \psi_{p_i}(n) \tag{15}$$

which is sparse in the Hermite transform domain with K = 35 components. The signal is corrupted by white Gaussian noise, with SNR=30 dB, and only 50% of samples are available. The amplitudes A_i of components and the orders of Hermite functions p_i are randomly chosen. The reconstruction is done using the proposed approach (Fig. 1), with MSE ~10^{-3}. We may observe that, although the signal is not strictly sparse, the exact reconstruction is achieved. Compared to the ℓ1-magic algorithm [8] which belongs to the group of convex optimizations as well, the processing time (in Matlab) for the proposed algorithm is 2 to 4 times lower (depending on the random measurements and amplitudes). Also, the proposed algorithm will progressively accelerate as the number of available samples increases.
The algorithm is examined also on a real QRS complex of the ECG signals [9], shown in Fig. 2. In order to achieve sparsity, QRS complex needs to be sampled at the points proportional to the roots of Hermite polynomial. The sparsification is done according to the procedure in [1] to obtain signal values at adequate position. The error due to sparsification is 5.23%, which is medically acceptable as long as it is less than 10% [1]. The reconstruction MSE is 1.262×10^{-7} obtained after 18 iterations of the proposed algorithm. The reconstructed results are compared with original signal in Fig. 2.

Conclusion: A gradient descent algorithm based on the Hermite transform domain solves the problem of CS reconstruction for signals that exhibit sparsity in the Hermite basis domain. The highly accurate algorithm performance is proven on both synthetic and real data.

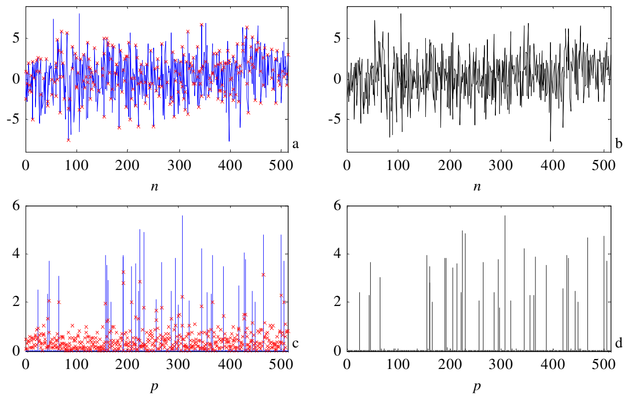

Fig. 1 *Reconstruction results: a - desired signal (solid line) and available samples (crosses), b - reconstructed signal; c – desired Hermite coefficients (solid line) and coefficients of signal with missing samples (crosses), d – Hermite coefficients of the reconstructed signal*

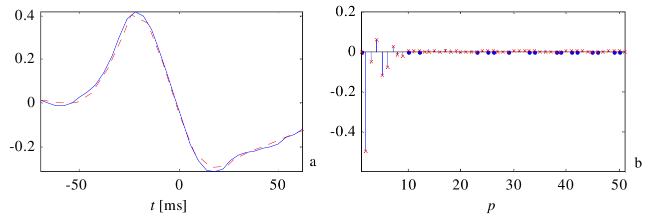

Fig. 2 *Reconstruction results for ECG signal (QRS complex): a - desired (solid line) and reconstructed signal (dashed line), b – Hermite coefficients of desired (solid line) and reconstructed signal (crosses)*

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