Analysis of Polynomial FM Signals Corrupted by Heavy-Tailed Noise

Braham Barkat, LJubiša Stanković

Abstract— In this paper, we consider the analysis of polynomial FM signals corrupted by additive heavy-tailed noise. Standard time-frequency techniques fail to analyze such signals. For that, we propose here a new technique, named the robust polynomial Wigner-Ville distribution (r-PWVD) to handle this case. We show that this representation outperforms the robust Wigner-Ville distribution (r-WVD) and the robust spectrogram in terms of artifacts suppression and high time-frequency resolution for this class of signals. Also, we show that the peak of the r-PWVD is an accurate instantaneous frequency estimator. Examples and Monte-Carlo simulations are presented in order to validate and prove the performance of the proposed algorithm.

I. INTRODUCTION

Time-frequency analysis has proved to be a powerful tool in the analysis of non-stationary signals, i.e., signals whose spectral contents vary with time [1]. Such signals can be found in many engineering applications such as radar, sonar and telecommunications.

In practice, the signal to be analyzed is always corrupted by noise. In general, and for various reasons, the corrupting noise is assumed to be additive and Gaussian. Analysis of non-stationary signals affected by additive Gaussian noise has been addressed in several places [2], [3], [4]. However, in some situations, the Gaussian assumption of the noise is not valid and, therefore, alternative analysis techniques are needed in this case.

Recently, a novel technique to analyze a sinusoid contaminated by additive noise having *unknown* heavy-tailed distribution was proposed in [5]. Examples of heavy-tailed distributions include Laplace, Cauchy and α-stable distributions with $\alpha < 2$. The use of these distributions have proved to be effective in modeling many real-life engineering problems such

as outliers and impulse signals [5]. The case of non-stationary signals was considered in [6], [7], [8].

In [6], [7], [8], the authors proposed the robust spectrogram and the robust Winger-Ville distribution (r-WVD) to address the problem of non-stationary signals embedded in heavy-tailed noise. However, it is known that the spectrogram suffers from low resolution in the time-frequency domain; while, the WVD suffers from the presence of artifacts for non-linearly frequency modulated (FM) signals and from cross-terms for multicomponent signals [1]. Moreover, and except for the pure sinusoid or the linear FM signal, the peak of the WVD is not exactly located at the signal instantaneous frequency (IF) but is always shifted from it [4]. Thus, it is important to have alternative high resolution techniques to deal with non-linear FM signals affected by impulsive noise.

In this paper, we consider the problem of polynomial FM signals corrupted by additive heavy-tailed noise. Since the polynomial WVD (PWVD) was designed to represent polynomial FM signals as a row of delta functions at the exact signal IF [9], it is the kernel of this particular representation that will be used in the design of the robust PWVD (r-PWVD). We show that the r-PWVD outperforms both the r-WVD and the robust spectrogram in the analysis of polynomial FM signals affected by heavy-tailed noise. We also show, using Monte-Carlo simulations, that the proposed r-PWVD is more accurate in estimating the IF of such noisy signals. We observe that some other works have considered parametric approaches in order to deal with impulsive noise. In particular, we can cite [15] where the authors proposed to estimate the phase coefficients of a chirp signal embedded in impulsive noise. Although a full comparision between

both techniques is out of the scope of this paper, we can state that the method proposed here applies to polynomial FM signals, and is not limited to linear FM signals only (which is the case of the method proposed in [15]).

The paper is organized as follows. In Section 2 we give a brief review of the robust spectrogram. In Section 3, we present the proposed r-PWVD algorithm. In the same section, we discuss the IF estimation and evaluate its statistical performance. Section 4 concludes the paper.

II. REVIEW OF THE ROBUST-SPECTROGRAM

Assume that the observed discrete-time noisy signal under consideration, $y(kT)$, can be written as $y(kT) = z(kT) + \epsilon(kT)$ where $z(kT) = A \exp\{j\phi(kT)\}$ is an arbitrary FM signal, $\epsilon(kT)$ is an additive noise, T is a sampling period and k an integer. Now, consider the optimization problem [6]

$$\hat{m} = \arg\min_{m} I(kT, f, m) \qquad (1)$$

with

$$I(kT, f, m) = \sum_{n=-N/2}^{N/2-1} w(nT)F[e(k, f, n)]$$

$$e(k, f, n) = y(kT + nT)e^{-j2\pi fnT} - m$$

where $w(nT)$ is a window function and m is an estimate of the expectation of the sample average of the quantity $y(kT + nT)e^{-j2\pi fnT}$ [6].

For the loss function $F(e) = |e|^2$, the optimum solution $\hat{m}$, obtained by solving $\partial I(t, f, m)/\partial m^* = 0$, yields the standard short-time Fourier transform (STFT)

$$STFT_y^s(kT, f) =$$

$$\sum_{n=-N/2}^{N/2-1} \frac{w(nT)}{\sum_{n=-N/2}^{N/2-1} w(nT)} y(kT + nT)e^{-j2\pi fnT}.$$
$$(2)$$

The standard spectrogram, or the squared magnitude of $STFT_y^s(kT, f)$, is the maximum likelihood (ML) estimate of the expectation in the case of Gaussian noise [6], but for non-Gaussian noise it produces poor results. In

this situation, and based on the minimax Huber's estimation theory, which is linked to the problem of spectra resistance to impulse noise, a better representation can be obtained by choosing $F(e) = |e|$. For the loss function $F(e) = |e|$, the optimal solution, called the robust STFT, is found to be [6]

$$STFT_y^r(kT, f)$$

$$= \sum_{n=-N/2}^{N/2-1} \frac{d(kT, f, n)}{\sum_{n=-N/2}^{N/2-1} d(k, f, n)}$$

$$\times y(kT + nT)e^{-j2\pi fnT}$$

where

$$d(k, f, n)$$

$$= \frac{w(nT)}{y(kT + nT)e^{-j2\pi fnT} - STFT_y^r(kT, f)}.$$
$$(3)$$

This robust STFT inherits the strong resistance property to the impulse noise, making it a very appropriate tool solution for the problem under investigation. Note that the right-hand side of the solution equation (3) also contains $STFT_y^r(kT, f)$. It suggests that the implementation of the robust spectrogram, the squared magnitude of the robust STFT, necessitates an appropriate iterative procedure (similar to the procedure for the robust PWVD outlined in Table I below). Also, note that the choice of the particular loss function $F(e) = |e|$ results from a minimax regression estimation problem detailed in [5]. In particular, the best loss function is obtained as the minimizer of the covariance matrix of the estimate. This loss function is different from the nonlinear function which is usually used to compress/suppress large samples of the noisy signal in the time domain [13], [14].

The time-frequency resolution of the robust spectrogram is not impressive [6]. In the sequel, we present techniques that can address the impulsive noise problem while maintaining high time-frequency resolution.

III. ROBUST PWVD

Here, we give a brief review of the PWVD and present the rational in the design of the r-PWVD to tackle the heavy-tailed noise prob-

lem. A statistical performance evaluation, using Monte-Carlo simulations, of the r-PWVD based IF estimation is also presented in this section.

A. Key Idea

The primary purpose of the window $w(nT)$ in the STFT (refer to the previous section) is to limit the extent of the signal to be transformed so that its spectral characteristics are reasonably stationary over the duration of the window. The more rapidly the signal characteristics change, the shorter the window should be (i.e., lower frequency resolution). On the other hand, increasing the window length results in a low time resolution. The best results, in terms of time-frequency resolution, can be obtained by using the optimal window length in the STFT. However, the optimal window length requires an a priori knowledge of the signal IF, which is difficult to obtain in general [1]. This limitation can be avoided by using the quadratic kernel of the WVD given by [1]

$$K_y(kT, nT)$$
$$= y(kT + 0.5nT) \cdot y^*(kT - 0.5nT). \quad (4)$$

The kernel $K_y(kT, nT)$ transforms, for each time instant kT, the IF of a monocomponent linear FM signal into a sinusoid at that particular time [4]. That is, the use of the above kernel guarantees stationarity of the signal for a larger extent of the signal (i.e., higher frequency resolution) compared to the STFT. Thus, using $K_y(kT, nT)$, instead of $y(kT + nT)$, in the optimization problem stated earlier results in the r-WVD [8]. This representation outperforms the robust spectrogram in terms of time-frequency resolution. Note that, an interpolation or appropriate oversampling is necessary in order to have non-integer values of the argument of the signal in Equation (4).

For higher-order polynomial FM signals given by

$$z(kT) = A \exp\{j \sum_{i=0}^{p} a_i(kT)^i\}, \quad (5)$$

the quadratic kernel of the WVD does not transform the IF into a pure sinusoid, at each

time instant kT, as it does for the linear FM case ($p = 2$). For such signals, the r-WVD produces some artifacts due to the mismatch between the signal polynomial order and the WVD [4] (see also Figure 2). To avoid this problem, and to improve the performance, we need to use a transform or a kernel that can map a higher-order polynomial FM signal, at each time instant kT, into a sinusoid. A transform that can accomplish this task is the PWVD kernel given by [9]

$$K_z^q(kT, nT)$$
$$= \prod_{i=1}^{q/2} z(kT + c_i nT) \cdot z^*(kT - c_i nT) \quad (6)$$

where the c_i are real coefficients, and q is an even integer number.

A general procedure to obtain the c_i and q for a fixed polynomial phase order p (refer to (5)), is outlined in [9]. For example, for $p = 2$, we have $q = 2$ and $c_1 = 0.5$. That's is, the WVD kernel in (4) is just a special case of the PWVD kernel. For quadratic ($p = 3$) or cubic ($p = 4$) FM signals, an appropriate kernel to use is the sixth order kernel given by [9]

$$K_z^{(6)}(kT, nT)$$
$$= [z(kT + 0.62nT) \, z^*(kT - 0.62nT)]$$
$$\times [z(kT + 0.75nT) \, z^*(kT - 0.75nT)]$$
$$\times [z(kT - 0.87nT) \, z^*(kT + 0.87nT)]. \quad (7)$$

Here, we should emphasise that, an interpolation is necessary in order to have signal values at non-integer argument points in Equation (6) (refer to [9] for more details). In what ensues, we will use the PWVD kernel, $K_z^{(q)}(kT, nT)$, in the design of the r-PWVD to analyze polynomial FM signals affected by impulsive noise.

B. Algorithm Development

Let us consider the optimization problem given by

$$\hat{m} = \arg \min_m J(kT, f, m) \quad (8)$$

where

$$J(kT, f, m) = \sum_{n=-N/2}^{N/2} w(nT) F[e(k, f, n)]$$

$$e(k, f, n) = K_y^{(q)}(kT, nT)e^{-j2\pi fnT} - m$$

with $K_y^{(q)}(kT, nT)$ being an appropriately designed kernel for the noisy polynomial FM signal $y(kT)$ under consideration.

If we choose the loss function as $F(e) = |e|^2$, we can show by solving for m the expression $\partial J(kT, f, m)/\partial m^* = 0$ that the optimal solution, labeled the standard PWVD, is equal to

$$W_y^s(kT, f) = \sum_{n=-N/2}^{N/2} \frac{w(nT)}{\sum_{n=-N/2}^{N/2} w(nT)}$$

$$\times K_y^{(q)}(kT, nT)e^{-j2\pi fnT}. \qquad (9)$$

Similarly to the standard spectrogram or standard WVD, the standard PWVD is not an adequate analysis tool in the presence of heavy-tailed noise, as shown in the example below. In the presence of such noise, we need to choose the loss function as $F(e) = |e|$. The choice of this particular function is well detailed and explained in [5], [10]. In this case, we find the optimal solution, labeled the robust PWVD (r-PWVD), to be

$$W_y^r(kT, f)$$

$$= \sum_{n=-N/2}^{N/2} \frac{d(k, f, n)}{D_0(kT, f)} K_y^{(q)}(kT, nT)e^{-j2\pi fnT}$$

$$(10)$$

$$d(k, f, n)$$

$$= \frac{w(nT)}{|K_y^{(q)}(kT, nT)e^{-j2\pi fnT} - W_y^r(kT, f)|}$$

$$(11)$$

$$D_0(kT, f) = \sum_{n=-N/2}^{N/2} d(k, f, n). \qquad (12)$$

Since, the quantity $W_y^r(kT, f)$ appears on the right as well as on the left hand side of the

Equation (10), an iterative procedure is necessary in order to obtain the r-PWVD. The algorithm of this representation is stated in Table I below.

Note that the above solution can also be written as

$$(W_y^r)^{i+1} = \mathcal{T}[(W_y^r)^i] = \frac{\sum_n f[(W_y^r)^i]\mathbf{x}_n}{\sum_n f[(W_y^r)^i]}$$

where

$$f[(W_y^r)^i] = \frac{w(nT)}{|K_y^{(q)}(kT, nT)e^{-j2\pi fnT} - (W_y^r)^i|}$$

and

$$\mathbf{x}_n = K_y^{(q)}(kT, nT)e^{-j2\pi fnT}.$$

For such forms, it was shown in [16] that if the initial value is within the interval $[\min(\mathbf{x}_n), \max(\mathbf{x}_n)]$, then the iterative algorithm will converge to a local minimum within the same interval. In our case, the function $J(kT, f, m)$ has a single minimum [16] and, therefore, the proposed iteration procedure will converge to that single (global) minimum since our initial value $W_y^{r0}(kT, f) = W_y^s(kT, f)$ is a mean value which satisfies the necessary condition of the convergence, namely, $W_y^{r0}(kT, f) \in [\min(\mathbf{x}_n), \max(\mathbf{x}_n)]$.

To check the validity and superiority of the proposed algorithm, let us consider the analysis of a quadratic FM whose frequency range is 0.1-0.4Hz (assume unit sampling period). Some corrupting noise is added to this noiseless signal. The additive noise $\epsilon(kT)$ consists of an α-stable process with skewness coefficient equal to zero and the dispersion coefficient equal to one. The standard PWVD, displayed in Figure 1 (left plot) yields a poor representation; while, the r-PWVD displayed in the same figure (right plot), clearly reveals the features of the noisy signal. Also, we compute the r-WVD of the same signal (Figure 2 (left plot)). The superiority of the r-PWVD over the r-WVD is obvious. For a closer comparison, slices of these two representations, taken at the middle of the time interval, are displayed in Figure 2. We see that the peak of the robust WVD is not located at the true IF of the signal (0.4 Hz in this case) but is shifted from it. In addition, the presence of

<div align="center">TABLE I</div>
<div align="center">ROBUST PWVD ALGORITHM</div>

1. Evaluate the standard PWVD using Equation (9).
2. For initialization purposes, set

$$W_y^{r0}(kT, f) = W_y^s(kT, f)$$

and the repetition index $i = 0$.
3. Set $i = i + 1$. Compute $d(k, f, n)$ using Equation (11).
4. Compute $D_0(kT, f)$ using Equation (12).
5. Evaluate the r-PWVD, for iteration i, $W_y^{ri}(kT, f)$ using Equation (10).
6. If the relative absolute difference between two iterations is smaller than a fixed threshold η, i.e.,

$$|W_y^{ri}(kT, f) - W_y^{r(i-1)}(kT, f)|/|W_y^{ri}(kT, f)| \leq \eta,$$

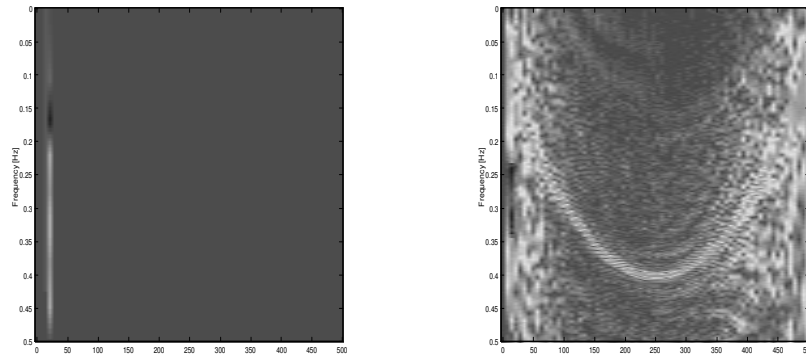then stop the algorithm. Otherwise go back to Step (3).

Fig. 1. The standard (left plot) and the robust (right plot) sixth order PWVD of a quadratic FM signal affected by impulsive noise.

the artifacts in the r-WVD tends to hide the real features of the signal.

C. IF Estimation

In many engineering applications, the IF characterizes important physical parameters of the signal [11]. Therefore, accurate and effective estimation of this quantity is of great importance. Based on the results of the previous sections, we see that the r-PWVD is well suited to estimate the IF of a polynomial FM signal affected by impulsive noise.

The Weierstrass theorem states that for a given function, defined and continuous on a closed interval, there exists a polynomial of a finite order p which is as "close" to the given function as desired [12]. The degree of the approximate polynomial, p, increases with increasing non-linearity of the approximated non-linear function. This result implies that the r-PWVD is very useful in the analysis, and IF estimation, of non-linear, not necessarily polynomial, FM signals corrupted by impulsive noise.

Here, we assess the statistical performance of the r-PWVD based IF estimator. Specifically, we evaluate, using Monte-Carlo simulations, the mean squared error of the IF estimate when the peak of the r-PWVD is used as an IF estimator. In addition, a statistical per-
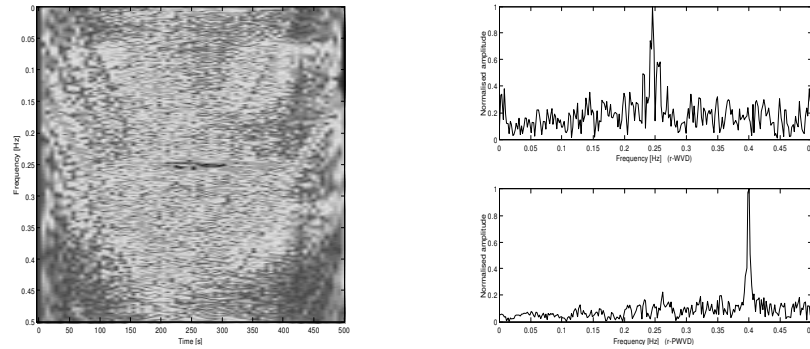
Fig. 2. The r-WVD (left plot) of the same signal displayed in Figure 1. The right plot displays slices, taken at the middle of the time interval, of the r-WVD (top plot) and the r-PWVD (bottom plot).
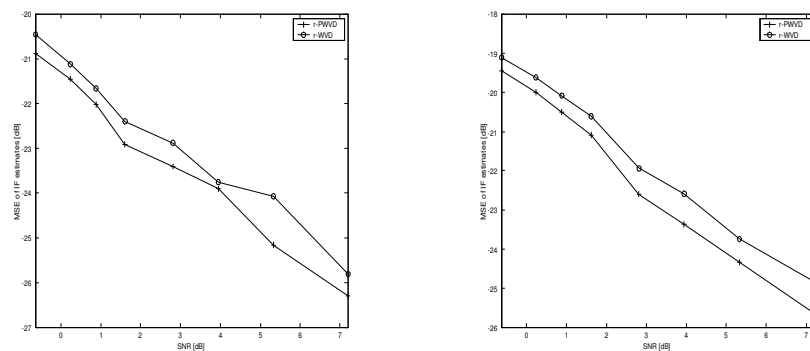

Fig. 3. M.S.E's of IF estimates, corresponding to the r-WVD ('o') and the r-PWVD6 ('+'), for a noisy quadratic FM signal (left plot) and a noisy cubic FM signal (right plot).

formance comparison with the r-WVD based IF estimator is also performed here. For that, let us consider a noisy quadratic FM signal $y(t)$ modeled as $y(t) = \exp\{j(a_1 t + a_2 t^2 + a_3 t^3)\} + a \cdot \epsilon(t)$ where a is a real parameter and the coefficients a_i, $i = 1, 2, 3$ are chosen so that the IF of the signal, $f_i(t) = \frac{1}{2\pi} d((a_1 t + a_2 t^2) + a_3 t^3)/dt = \frac{1}{2\pi}(a_1 + 2a_2 t + 3a_3 t^2)$, is confined in the range 0.1-0.4Hz (assume a unit sampling interval). The complex noise $\epsilon(t)$ consists of independent real and imaginary parts. The samples of each part are chosen to be independent and identically distributed (i.i.d), taken from an α-stable process with a zero skewness coefficient and a unit dispersion coefficient. In the experiment, we fix the signal length equal to $N = 501$ samples and the window length, used in the r-PWVD implementation, equal to 101 samples. The r-WVD and the robust sixth order PWVD (r-PWVD6) of one realization have already been displayed in Figures 1 and 2.

We estimate the IF of the noisy signal, at the middle of the time intrval (corresponding to the exact IF: 0.4 Hz), as the argument of the peak of the robust representations. We repeat the estimation process 6000 times for each value of the signal-to-noise ratio (SNR). In Figure 3 (left plot), we plot the estimates mean squared errors (M.S.E's) versus the signal-to-noise ratio (in dB) for both r-WVD and r-PWVD6. We re-run the same experiment for a cubic FM signal (same figure (right plot)). The accuracy and superiority of the r-PWVD6 over the r-WVD is evident in both examples.

IV. Conclusion

In this paper, we proposed a new technique to analyze polynomial FM signals corrupted by additive impulsive noise. We showed that the proposed technique, referred to as r-PWVD, outperforms the robust Wigner-Ville distribution (r-WVD) and the robust spectrogram in terms of artifacts suppression and high time-frequency resolution. In addition, Monte-Carlo simulations have shown that the r-PWVD is very accurate in estimating the instantaneous frequency of the noisy signal and its statistical performance is superior to that of the r-WVD.

V. Acknoledgement

References

[1] L. Cohen. Time-Frequency Analysis, Prentice-Hall (1995).

[2] M. Amin. Time-Frequency Signal Analysis: Methods and Applications, chapter Time-Frequency Spectrum Analysis and Estimation for Non-Stationary Random Processes, Longman-Chesire (1992).

[3] P. Rao, F.J. Taylor. Estimation of IF using the discrete Wigner-Ville Distribution, Electronics Letters 26 (1990) 246–248.

[4] B. Barkat, B. Boashash. Instantaneous Frequency Estimation of Polynomial FM Signals Using the Peak of the PWVD: Statistical Performance in the Presence of Additive Gaussian Noise, IEEE Trans. on Signal Processing 47 (9) (Sept. 1999) 2480–2490.

[5] V. Katkovnik. Robust M-periodogram, IEEE Trans. on Signal Processing 46 (11) (Nov. 1998) 3104–3109.

[6] V. Katkovnik, I. Djurović, LJ. Stanković. Time-Frequency Signal Analysis, chapter Robust time-frequency representations, Prentice-Hall 2002. (In print).

[7] I. Durović, LJ. Stanković, V. Katkovnik. Median filter realization of the robust time-frequency distributions, Signal Processing 81 (8) (Aug. 2001) 1771–1776.

[8] I. Durović, LJ. Stanković. Robust Wigner distribution with application to the instantaneous frequency estimation, IEEE Trans. on Signal Processing 49 (12) (Dec. 2001) 2985–2993.

[9] B. Barkat, B. Boashash. Design of Higher-Order Polynomial Wigner-Ville Distributions, IEEE Trans. on Signal Processing 47 (9) (Sept. 1999) 2608–2611.

[10] I. Durović, LJ. Stanković, J.F. Böhme. Estimates of the Wigner distribution in Gaussian noise environment, AEÜ- Int. Journal Elec. Comm., 56 (2) (2002).

[11] D. König, J. F. Böhme. Wigner-Ville Spectral Analysis of Automotive Signals Captured at Knock. Applied Signal Processing 3 (1996) 54–64.

[12] R.L. Burden, J.D. Faires, A.C. Reynolds. Numerical Analysis, Prindle, Weber & Schmidt, Boston, U.S.A (1978).

[13] S.R. Kim, A. Efros. Adaptive robust impulse noise filtering, IEEE Trans. on Signal Processing 43 (8) (Aug. 1995) 1855-1866.

[14] S.M. Kay. Fundamentals of Statistical Signal Processing: Detection Theory, Printics-Hall, U.S.A (1998).

[15] F. Gini, M. Montanari, L. Verrazzani. Estimation of chirp radar signals in compound-Gaussian clutter: a cyclostationary approach, IEEE Trans. on Signal Processing 48 (4) (Apr. 2000) 1029-1039.

[16] S.Kalluri, G.R. Arce. Fast algorithms for weighted myriad algorithm computation by fixed point search. IEEE Trans. on Signal Processing 48 (1) (Jan. 2000) 159-171.