Modifed Wigner Bispectrum and its Generalizations

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Abstract— **Wigner bispectrum of multicomponent signals is studied. Its modified and reduced forms are introduced. A generalization of the presented forms to the Wigner higher order spectra (WHOS), in the case of multicomponent signals, is provided. From our previous work it is known that the cross terms removal (reduction) is possible for an odd-order spectra with equal number of conjugated and nonconjugated terms. Here, we extended the analysis to an even-order spectra, as well. The theory is illustrated by examples.**

I. INTRODUCTION

Higher order spectra have been intensively studied lately. We refer the reader to the review papers [1], [2]. Using the concept of higher order spectra, the definitions of time-varying higher order spectra for the case of nonstationary signals, with special attention devoted to the Wigner higher order spectra (WHOS), are introduced, [3], [4]. The analysis of the WHOS of multicomponent signals is performed in [5], where it is shown that the separation of auto terms and cross terms may be easily done in the case of an odd-order spectra. That analysis led to the definition of the L-Wigner distribution, as a special and reduced form of the Wigner higher order spectra, [6], [7].

In this paper we show that, by an appropriate transformation of the signal before the application of the WHOS, the separation of the auto terms and cross terms is possible for an arbitrary order of the WHOS. In the first part of the paper, the Wigner bispectrum, as the lowest one that preserves the phase information, is studied in details. Its analysis and modifications provided a basis for the generalizations in the second part of the paper.

Circuits Systems and Signal Processing, vol.16, No.1, Jan. 1997.

II. BISPECTRUM AND WIGNER BISPECTRUM

Second-order cumulant of a random zero-mean signal $x(t)$ is identical to its autocorrelation function (the second order moment), and it is defined by

$$c_2^x(\tau) = m_2^x(\tau) = E\left\{x^*(t)x(t+\tau)\right\} \quad (1)$$

The Fourier transform of the autocorrelation function is the power spectrum:

$$S_2(\omega) = F\left\{c_2^x(\tau)\right\} = \int_\tau c_2^x(\tau)e^{-j\omega\tau}d\tau \quad (2)$$

The previous relations may be extended to the higher order statistics [1], [2]. The third order moment $m_3^x(\tau_1,\tau_2)$ is given by:

$$m_3^x(\tau_1,\tau_2) = E\left\{x^*(t)x(t+\tau_1)x(t+\tau_2)\right\}$$

while the third order cumulant is defined in terms of the first, second and third order moment as:

$$c_3^x(\tau_1,\tau_2) = m_3^x(\tau_1,\tau_2) -$$
$$m_1^x\left[m_2^x(\tau_1)+m_2^x(\tau_2)+m_2^x(\tau_1-\tau_2)\right] + 2(m_1^x)^3 \quad (3)$$

where m_1^x represents the mean value of the signal. Without loss of generality, we have assumed that $m_1^x = 0$, when $c_3^x(\tau_1,\tau_2) = m_3^x(\tau_1,\tau_2)$.

The Fourier transform of $c_3^x(\tau_1,\tau_2)$ is called bispectrum:

$$B(\omega_1,\omega_2) = \int_{\tau_1}\int_{\tau_2} c_3^x(\tau_1,\tau_2)$$
$$\times e^{-j(\omega_1\tau_1+\omega_2\tau_2)}d\tau_1 d\tau_2 \quad (4)$$

where

$$c_3^x(\tau_1,\tau_2) =$$
$$E\left\{x^*(t+\alpha)x(t+\tau_1+\alpha)x(t+\tau_2+\alpha)\right\}$$

and α is an arbitrary constant.

For deterministic nonstationary signals, replacing $E\{x^*(t+\alpha)x(t+\tau_1+\alpha)x(t+\tau_2+\alpha)\}$ by $x^*(t+\alpha)x(t+\tau_1+\alpha)x(t+\tau_2+\alpha)$ we arrive at the Wigner bispectrum (WB):

$$WB(t,\omega_1,\omega_2) = \int_{\tau_1}\int_{\tau_2} x^*(t+\alpha)x(t+\tau_1+\alpha)$$
$$\times\, x(t+\tau_2+\alpha)e^{-j(\omega_1\tau_1+\omega_2\tau_2)}d\tau_1 d\tau_2 \quad (5)$$

where the value of $\alpha = -\frac{\tau_1}{3} - \frac{\tau_2}{3}$ is chosen such that the mean value of the signal's arguments in the above integral is equal to t, [3], [4].

III. WIGNER BISPECTRUM OF THE MULTICOMPONENT SIGNALS

The WB of a signal $x(t)$ is introduced by Gerr, [3]. Fonolosa and Nikias, [4], have extended this idea to the higher order time-varying spectra, and provided the detailed study of the WB itself. Its definition (5), with $\alpha = -\frac{\tau_1}{3} - \frac{\tau_2}{3}$, is:

$$WB(t,\omega_1,\omega_2) =$$
$$\int_{\tau_1}\int_{\tau_2} x^*(t-\frac{\tau_1}{3}-\frac{\tau_2}{3})x(t+\frac{2\tau_1}{3}-\frac{\tau_2}{3})$$
$$\times\, x(t-\frac{\tau_1}{3}+\frac{2\tau_2}{3})e^{-j\omega_1\tau_1-j\omega_2\tau_2}d\tau_1 d\tau_2 \quad (6)$$

Using the Fourier transform $X(\omega)$ of a signal $x(t)$, the previous expression may be written as:

$$WB(t,\omega_1,\omega_2) = \frac{1}{2\pi}\int_{\theta} X^*(\omega_1+\omega_2+\frac{\theta}{3})$$
$$\times\, X(\omega_1-\frac{\theta}{3})X(\omega_2-\frac{\theta}{3})e^{-j\theta t}d\theta \quad (7)$$

The above definitions exhibit apparent nonlinearity of the WB. Among other effects, this WB's nonlinearity manifests itself through cross-terms, when it is used for time-frequency analysis of multicomponent signals. As an illustration, let us consider an M-component signal

$$x(t) = \sum_{i=1}^{M} x_i(t). \quad (8)$$

Its WB has the form:

$$WB(t,\omega_1,\omega_2) =$$
$$\sum_{i_1=1}^{M}\sum_{i_2=1}^{M}\sum_{i_3=1}^{M}\int_{\tau_1}\int_{\tau_2} x^*_{i_1}(t-\frac{\tau_1}{3}-\frac{\tau_2}{3})$$
$$\times\, x_{i_2}(t+\frac{2\tau_1}{3}-\frac{\tau_2}{3})x_{i_3}(t-\frac{\tau_1}{3}+\frac{2\tau_2}{3})$$
$$\times\, e^{-j\omega_1\tau_1-j\omega_2\tau_2}d\tau_1 d\tau_2. \quad (9)$$

It is obvious that the WB has M^3 terms, where only M of them ($i_1 = i_2 = i_3$) are auto terms, while the remaining $M^3 - M$ are the cross terms. This clearly indicates the difficulties arising in time-frequency analysis of multicomponent signals when using the WB by definition (6).

Our aim is to provide a modified WB which will, under certain conditions, produce cross-terms free time-frequency representation of a multicomponent signal.

In order to provide an initial insight into the problem, consider a very simple example of a sum of two complex exponential signals:

$$x(t) = e^{j(t\omega_{01}+\varphi_1)} + e^{j(t\omega_{02}+\varphi_2)} \quad (10)$$

whose Fourier transform is:

$$X(\omega) = 2\pi e^{j\varphi_1}\delta(\omega-\omega_{01}) + 2\pi e^{j\varphi_2}\delta(\omega-\omega_{02}) \quad (11)$$

The illustration of integrand in the WB, eq. (7), for $\theta = 0$, is presented in Fig. 1. The WB is different from zero for:

$$\omega_1+\omega_2+\frac{\theta}{3}-\omega_{0i} = 0$$
$$\omega_1-\frac{\theta}{3}-\omega_{0j} = 0$$
$$\omega_2-\frac{\theta}{3}-\omega_{0k} = 0 \quad (12)$$

where i,j,k may assume values of 1 or 2.

For $i = j = k$ the auto terms are obtained, and their locations are defined by $\theta = -\omega_{0i}$, $(\omega_1,\omega_2) = \frac{2}{3}(\omega_{0i},\omega_{0i})$. All the remaining solutions of the above system are associated with the cross terms.

Note that auto terms are distributed along the symetral $s: \omega_1 = \omega_2$ in the (ω_1,ω_2) space. This is an important conclusion which enables one to eliminate all the terms that do not lie on this line. But, besides auto terms, this line contains some cross terms, as well. The cross

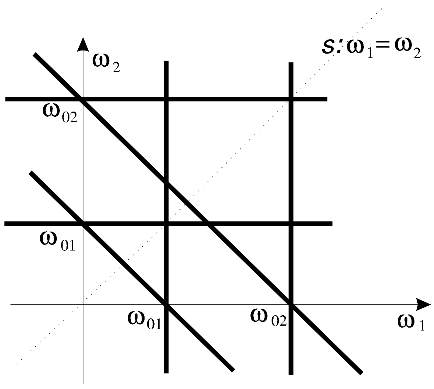

Fig. 1. Illustration of the integrand in Wigner bispectrum for $\theta = 0$.

terms lying on line s are obtained from system (12) with $j = k \neq i$. The cross terms' locations are at $\theta = \omega_{0i} - 2\omega_{oj}$. It is evident that, for an arbitrary two-component (multicomponent) signal, we may not distinguish auto and cross terms lying on the line s. If we, however, suppose that we somehow manage to have $\omega_{02} = 2\omega_{01}$, then the cross term will be, for any ω_{01}, positioned at $\theta = 0$, and will lie at $(\omega_1, \omega_2) = (\omega_{01}, \omega_{01})$, what is the natural auto term position of the first component. All the other components will be dislocated from $\theta = 0$ and may be easily eliminated using a window in $\theta-$domain. This is the basic idea which will be used in the definition of the modified WB.

IV. REDUCED AND MODIFIED WIGNER BISPECTRUM

From the previous analysis we have concluded that all the auto terms, as well as some cross terms, are positioned on the line $s : \omega_1 = \omega_2 = \omega$. The WB on this line is defined by:

$$RWB(t, \omega, \omega) =$$

$$\frac{1}{2\pi} \int_\theta X^*(2\omega + \frac{\theta}{3}) X^2(\omega - \frac{\theta}{3}) e^{-j\theta t} d\theta \quad (13)$$

and will be referred to as the reduced Wigner bispectrum (RWB).

Following the idea presented at the end of Section 3, if one associates to each signal component another component having its doubled frequency, then the desired terms will lie at the θ origin. Its position along ω axis will be at the positions equal to the signal components' frequencies, Fig.2. These will be exactly the terms we look for. Note that we have used the notion of desired and not the auto term, since these are not the auto terms in the conventional sense. All the other terms may be removed using a frequency domain window $P(\theta)$ which will be of a low pass type. Including the window $P(\theta)$ in the expression for the reduced WB, we obtain the modified Wigner bispectum (MWB):

$$MWB(t, \omega, \omega) =$$

$$\frac{1}{2\pi} \int_\theta P(\theta) X_e^*(2\omega + \frac{\theta}{3}) X_e^2(\omega - \frac{\theta}{3}) e^{-j\theta t} d\theta \quad (14)$$

As it has been mentioned, for the MWB one needs to modify the input signal. The modification is performed in order to associate each signal's component with the corresponding one having doubled original component frequency. The simplest way to achieve this is to form a signal [8], [9]:

$$X_e(\omega) = X(\omega) + X(\frac{\omega}{2})$$

or

$$x_e(t) = x(t) + 2x(2t) \quad (15)$$

Note that a similar effect may be achieved for asymptotic signals [10] by using the transformation $x_e(t) = x(t) + x^2(t)$.

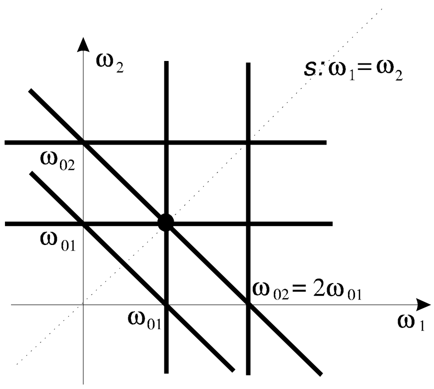

Fig. 2. Coupling of signal components

For the considered two-component signal (10), the MWB of a signal modified by (15), has the form

$$MWB(t,\omega) =$$
$$k_1\delta^2(\omega-\omega_{01})e^{j\varphi_1} + k_2\delta^2(\omega-\omega_{02})e^{j\varphi_2} \quad (16)$$

Observe that $MWB(t,\omega)$ has only the terms at the signal components instantaneous frequencies, as well as that **the phase information is preserved in the MWB**, which is not the case in the widely used quadratic time-frequency distributions.

A time domain definition of the MWB is:

$$MWB(t,\omega) =$$
$$p(t) *_t \int_{\tau_1}\int_{\tau_2} x^*(t-\frac{\tau_1}{3}-\frac{\tau_2}{3})x(t+\frac{2\tau_1}{3}-\frac{\tau_2}{3})$$
$$\times x(t-\frac{\tau_1}{3}+\frac{2\tau_2}{3})e^{-j\omega(\tau_1+\tau_2)}d\tau_1 d\tau_2 \quad (17)$$

where $*_t$ denotes a convolution in time, while $p(t)$ is the inverse Fourier transform of $P(\theta)$. From expression (17) one may conclude that for the noisy signals the MWB is unbiased, what is a significant difference when compared to the quadratic distributions, [15], [16].

Observe that the previous development was done for signals with constant frequency in a considered time interval. Generalization to the more complex signal forms, and for an arbitrary order of time-varying spectra, will be presented in Section 6. A numerical example with nonstationary signal is presented in the next Section.

V. NUMERICAL EXAMPLE

Let us consider a multicomponent nonstationary signal defined as:

$$x(t) = e^{j10\pi t}\Pi\left(\frac{t+0.125}{0.25}\right) +$$
$$+e^{-j8\pi t}\Pi\left(\frac{t-0.125}{0.25}\right),$$

where

$$\Pi(t) = \begin{cases} 1 & |t| < 1 \\ 0 & elsewhere \end{cases}$$

whose real and imaginary parts are shown in Fig.3a. The motivation for this particular form of signal was found in the electromagnetic diffraction theory where the electrical field, scattered from the truncated arrays, corresponds to the above signal form, [12], [13], [14].

Applying the modified signal:

$$x_e(t) = x(t) + x^2(t)$$

to the MWB (14) with a rectangular window $P(\theta)$ having the unity value for $|\theta/3| \leq 2\pi$, we got the distribution shown in Fig.3d. The full (three-dimensional) Wigner bispectrum (without $P(\theta)$ window) is shown in Fig.3b). The case with a rectangular window $P(\theta)$ in (7) (having the unity value for $|\theta/3| \leq 4\pi$) is presented in Fig.3c). The advantages of the MWB (over the Wigner distribution calculated by definition (7)), as a tool for time-frequency

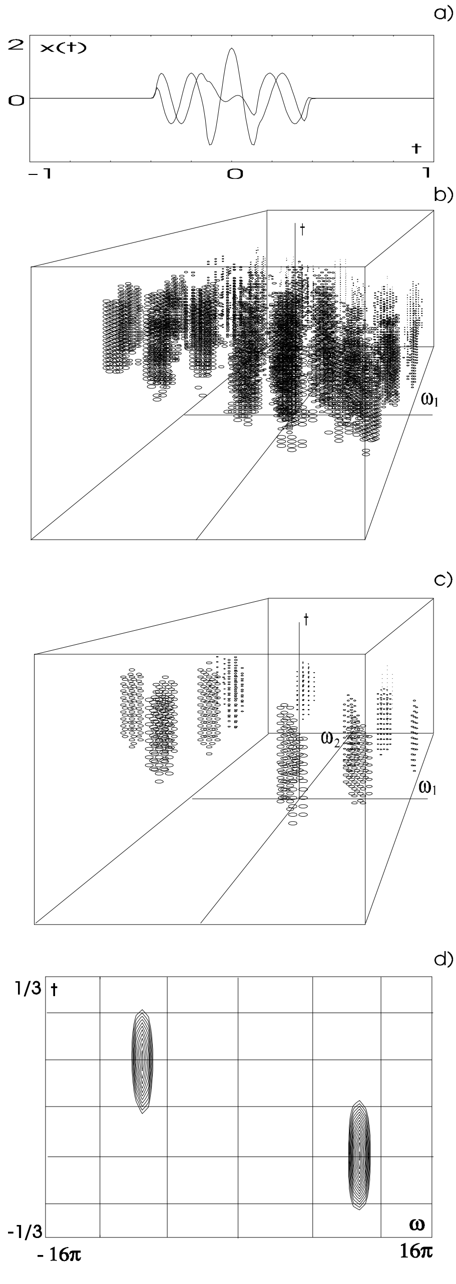

Fig. 3. a) Multicomponent nonstationary signal (its real and imaginary parts), b) Wigner bispectrum, c) Wigner bispectrum with the window $P(\theta)$ of the width $W_p = 4\pi$, d) Modified Wigner bispectrum with the window $P(\theta)$ of the width $W_p = 2\pi$.

analysis, are evident from Fig.3. Similar forms, with the preservation of phase information, may be obtained using signal transformations defined by (15).

VI. GENERALIZATIONS

The Wigner higher order spectra of a deterministic signal $x(t)$ are defined by [4]:

$$W_k(t,\omega_1,\omega_2,...,\omega_k) =$$

$$\int_{\tau_1}\int_{\tau_2}...\int_{\tau_k} x^*(t-\alpha)\prod_{i=1}^{L-1} x^*(t-\alpha+\tau_i)$$

$$\times\prod_{i=L}^{k} x(t-\alpha+\tau_i)\prod_{i=1}^{k} e^{-j\omega_i\tau_i}d\tau_i \qquad (18)$$

where:

$$\alpha = \frac{1}{k+1}\sum_{i=1}^{k}\tau_i$$

In terms of the Fourier transform $X(\omega)$, the above equation becomes:

$$W_k(t,\omega_1,\omega_2,...,\omega_k) =$$

$$\frac{1}{2\pi}\int_\theta X^*(\sum_{i=1}^{k}\omega_i+\frac{\theta}{k+1})$$

$$\times\prod_{i=1}^{L-1} X^*(-\omega_i+\frac{\theta}{k+1})$$

$$\times\prod_{i=L}^{k} X(\omega_i-\frac{\theta}{k+1})e^{-j\theta t}d\theta \qquad (19)$$

In order to analyze properties and locations of the auto-terms and cross-terms, let us consider the multicomponent signal whose Fourier transform may be expressed as:

$$X(\omega) = \sum_{m=1}^{M} X_m(\omega-a_m) \qquad (20)$$

where $X_m(\omega) = 0$, for $|\omega| \geq W$, and $a_m = constant$[1].

For signal (20), the WHOS becomes:

$$W_k(t,\omega_1,\omega_2,...,\omega_k) = \sum_{m_1=1}^{M}\sum_{m_2=1}^{M}...\sum_{m_{k+1}=1}^{M}$$

$$\int_\theta X^*_{m_1}(\sum_{i=1}^{k}\omega_i+\frac{\theta}{k+1}-a_{m_1})\times$$

$$\times\prod_{i=1}^{L-1} X^*_{m_{i+1}}(-\omega_i+\frac{\theta}{k+1}-a_{m_{i+1}})$$

$$\times\prod_{i=L}^{k} X_{m_{i+1}}(\omega_i-\frac{\theta}{k+1}-a_{m_{i+1}})e^{-j\theta t}d\theta \quad (21)$$

For multicomponent signals the definition of WHOS implies that the effects of nonlinearity give rise to the heavy presence of the cross-terms. Note, for example, that the Wigner k-spectrum of an M-component signal contains M^{k+1} terms, only M of them being auto-terms (for example, in the Wigner trispectrum of a four-component signal there are 256 terms, with 252 cross terms). This effectively illustrates the problem of emphatic cross-terms in the WHOS.

A. Positions of Terms in the ω Space

The integrand in (21) is different from zero if the following inequalities hold:

$$\left|\sum_{i=1}^{k}\omega_i+\frac{\theta}{k+1}-a_{m_1}\right| < W;$$

$$\left|-\omega_j+\frac{\theta}{k+1}-a_{m_{j+1}}\right| < W;$$

$$\left|\omega_l-\frac{\theta}{k+1}-a_{m_{l+1}}\right| < W$$

$$j=1,...,L-1, \quad l=L,...,k. \qquad (22)$$

[1]Note that the signal

$$x(t) = \sum_{m=1}^{M} r_m(t)e^{j\phi_m(t)}$$

may be written in the previous form if all its components belong to the class of asymptotic signals, i.e. the amplitude variations are much slower than the phase variations, [10,17]. In addition, we will assume that the instantaneous frequencies $\phi'_m(t)$ may be treated as constants a_m, within the considered time interval and $X_m(\omega) = FT\{r_m(t)\}$.

Eliminating θ from (22), the terms' positions along ω_1 , $\omega_2, ..., \omega_k$ follow:

$$\left| -\omega_j + \frac{\sum_{i=1}^{L} a_{m_i} - \sum_{i=L+1}^{k+1} a_{m_i} - (k+1)a_{m_{j+1}}}{k+1} \right| < \frac{2k}{k+1}W \ , \quad j = 1, 2, ..., L-1$$

$$\left| \omega_l - \frac{\sum_{i=1}^{L} a_{m_i} - \sum_{i=L+1}^{k+1} a_{m_i} + (k+1)a_{m_{j+1}}}{k+1} \right| < \frac{2k}{k+1}W, \quad l = L, L+1, ..., k \tag{23}$$

Bearing in mind that an auto term is obtained for $m_1 = m_2 = ... = m_{k+1} \equiv m$, we get the auto terms' positions as:

$$\left| -\omega_j + \frac{(2L-k-1)a_m}{k+1} - a_m \right| < \frac{2k}{k+1}W \ ,$$

$$\left| \omega_l - \frac{(2L-k-1)a_m}{k+1} - a_m \right| < \frac{2k}{k+1}W$$

where $j = 1, 2, .., L-1$ and $l = L, ..., k$. Observe that the auto terms' centers are located along the line:

$$s: \ Q\,\omega_1 = Q\omega_2 = ... = Q\omega_{L-1} = \\ = q\omega_L = ... = q\omega_k = \omega \tag{24}$$

where Q= $(k+1)/(2L-2k-2)$ and $q = (k+1)/2L$. The auto terms lie at the point $\omega = a_m$.

Generally, the auto terms' positions are different along different axis and they are biased with respect to the a_m (the natural auto term position). Two especially interesting cases are: 1) For $L = 1$, the positions are axis independent, [4]. They are determined by $\omega_i = \frac{2}{k+1}a_m$, $i = 1, 2, ..., k$. In this case, line s is the symmetral of the first quadrant in ω space:

$$s: \ \omega_1 = \omega_2 = ... = \omega_k = \omega \tag{25}$$

2) Another interesting case is $L = (k+1)/2$ when the auto terms' positions are at $+a_m$ or $-a_m$. In this case $Q = -1$, $q = 1$ so the auto terms lie along the line:

$$s: \ -\omega_1 = -\omega_2 = ... = -\omega_{L-1=} \\ = \omega_L = ... = \omega_k = \omega \tag{26}$$

at their natural positions $\omega = a_m$, [5], [6].

All the terms whose centers lie outside line s, defined by (24), are the cross-terms. This is an important result, since the significant number of cross terms may be eliminated from further consideration. But, besides the auto-terms, line s contains certain number of cross-terms. The elimination of the remaining cross terms may be only based on their locations along θ axis in (21).

B. Positions of Terms Along θ Direction

For the multicomponent signals it is of great importance to investigate the terms' (auto terms and cross terms) locations along θ axis in (21). Namely, in the ordinary Wigner analysis it is shown that this $\theta-$dimension may provide an efficient basis for the auto terms and cross terms separation [6], [11], [17]. In Section 4, it is extended to the WB. The same idea will lead to the method for the cross terms suppression in the Wigner higher order spectra.

The terms' positions along θ axis may be obtained from (22) as:

$$\left| \theta - \sum_{i=1}^{L} a_{m_i} + \sum_{i=L+1}^{k+1} a_{m_i} \right| < (k+1)W \tag{27}$$

Especially interesting are the cases when the auto terms' positions along θ are signal independent.

Theorem 1. **The auto term position along θ axis is signal independent only if $L = (k+1)/2$.**

Proof. For an auto term $a_{m_i} \equiv a$ holds, so the relation (27) becomes:

$$|\theta - (k-2L+1)a| < (k+1)W$$

and the proof immediately follows. *QED*

In this case the auto terms' positions are signal independent and they are located at (and around) θ axis origin, while the cross terms are dislocated from the θ origin. This fact may be used for the auto terms and cross terms separation. It is studied in details in [5], [6]. This case may be considered as a generalized energy distribution. Note that, in this case, the information about the phase is lost.

If one wants to preserve the information about the phase, then the value $L = (k+1)/2$ must not be used. But, if $L \neq (k+1)/2$ then the auto terms' positions are signal dependent, which prevents their separation from the cross terms.

However, an interesting trick may be used in order to extract the information about the auto terms. It will be shown that it is possible to modify the signal so that the cross terms appear at the "natural" auto terms positions $\omega_i = a_{m_i}$, $i = 1, 2, ..., k$. Further analysis will be based on the theorems that follow.

Theorem 2. The Wigner k-spectrum of a two-component signal will have terms centered at the θ origin, if their frequencies satisfy:

$$a_1 = \left(1 + \frac{k - 2L + 1}{m - n}\right) a_2 \qquad (28)$$

where m and n are arbitrary integers such that $0 \leq m \leq L$ and $0 \leq n \leq k - L + 1$, $(m \neq n)$.

Proof: Note that in the case of two-component signals the instantaneous frequencies a_{m_i} in the general relation (27) may assume values a_1 or a_2. Let the number of appearances of a_1 in the first sum in (27) is m $(0 \leq m \leq L)$, while in the second sum it appears n times $(0 \leq n \leq k - L + 1)$. Under these general assumptions, relation (27) takes the following form:

$$|\theta - (m - n)a_1 + (m - n + k - 2L + 1)a_2| <$$
$$< (k + 1)W$$

thus, for $\theta = 0$, directly producing the proof of the theorem. *QED*

Theorem 3. If, in the Wigner k-spectrum of a multicomponent signal (20), each signal component having the frequency a_m, is accompanied with the corresponding one having the frequency $(k - 2L + 2)a_m$, then there will exist terms centered at the θ origin, with positions along the line:

$$s: \quad \omega_1 = \omega_2 = .. = \omega_{L-1} =$$
$$= -\omega_L = .. = -\omega_k = -\omega = -a_m \qquad (29)$$

at the points $\omega = a_m$.

Proof: From (22) follows that the terms are on line s for $\theta = 0$ iff $m_2 = m_3 = ... = m_{k+1}$. In the sense of Theorem 2, that means $m = 1$, $n = 0$, and the proof directly follows. Q.E.D.

Note that terms lying at the positions defined by Theorem 3 (natural auto terms' positions) are actually the cross terms. In this way, the Wigner k-spectrum has the terms at the positions $\omega = a_m$ $(m = 1, 2, ..., M)$. At the same time, these terms are obtained by the integration around θ axis origin, i.e. their positions along θ are not signal dependent.

C. Modifications in the Wigner k-Spectrum

Now, we have to provide a way to generate a signal whose each component will have the associated component, with frequency $p = (k - 2L + 2)$ times greater than the original one (according to Theorem 3). One way to achieve this is by creating a signal of the form:

$$x_e(t) = x(t) + x^p(t) \qquad (30)$$

The case of $k = 2$, was treated in Section 2. This approach works out well for monocomponent signals. But, since the primary goal of this paper is the analysis of multicomponent signals, then one should note the shortcoming stemming from the appearance of additional undesirable terms that may result from the $p-$th power of a multicomponent signal[2].

The above shortcoming may be avoided for the signals defined in footnote 1, at the same time satisfying the conditions of Theorem 3, if one resorts to the frequency domain, by defining the signal in the form [8], [9]:

$$X_e(\omega) = X(\omega) + X(\omega/p) \qquad (31)$$

Its time domain equivalent is:

$$x_e(t) = x(t) + px(pt) \qquad (32)$$

Raising the signal (in (30)) to the p-th power may introduce additional terms, while

[2] In the case $p = 2$, additional terms in the modified Wigner higher order spectra may appear if there exist such components that $a_m + a_n = 2(a_l + a_k)$ for any $m, n, k, l = 1, 2, ..., M$. These terms will be located along ω at $\omega = a_m + a_n$.

the presence of p in (31) increases the complexity of numerical realization. So, regardless of the accepted signal modification, it is of utmost importance to keep p as low as possible. Generally, one may distinguish two cases:

-If k is an odd integer (Wigner distribution, $k = 1$, and Wigner trispectrum, $k = 3$, are the lowest ones in this group), then the minimal value $p = 0$ is obtained with $L = (k+1)/2$. In this case no signal modification is needed.

-If k is an even integer (Wigner bispectrum, $k = 2$, is the lowest one in this group), **the minimal value of p satisfying the previous condition is $p = 2$, for any order of k, with $L = k/2$. Thus, the modification for any even order spectra reduces to the modification needed in the WB, eq.(15).**

D. *Modified Versions of the Wigner k-Spectrum*

From the previous analysis the modified version of the Wigner k-spectrum may be defined, considering only line s. The window $P(\theta)$ is used to control the cross-terms' presence along s. The modified Wigner k-spectrum is defined by:

$$MW_k(t,\omega) = \int_\theta P(\theta) X^*((k-2L+2)\omega + \frac{\theta}{k+1}) \times X^{*(L-1)}(\omega + \frac{\theta}{k+1}) \times X^{k-L+1}(\omega - \frac{\theta}{k+1}) e^{-j\theta t} d\theta \qquad (33)$$

The integration over "auto terms" (in the sense of Theorem 3) is completely performed, while the other terms are eliminated, if the window $P(\theta)$ width W_p ($P(\theta) = 0$ for $\theta > W_p$) satisfies:

$$(k+1)W < W_P < \min_{m,n} \{|a_m - (k-2L+2)a_n|\} - (k+1)W \qquad (34)$$

This relation follows from (27) considering only line (29), ($m = m_1, n = m_2 = m_3 = ... = m_{k+1}$).

The special cases are:

1^o Modified Wigner bispectrum with $L = k/2 = 1$:

$$MW_2(t,\omega) = \int_\theta P(\theta) X^*(2\omega + \frac{\theta}{3}) X^2(\omega - \frac{\theta}{3}) e^{-j\theta t} d\theta \qquad (35)$$

2^o Modified Wigner trispectrum with $L = (k-1)/2 = 1$:

$$MW_3(t,\omega) = \int_\theta P(\theta) X^*(3\omega + \frac{\theta}{4}) X^3(\omega - \frac{\theta}{4}) e^{-j\theta t} d\theta \qquad (36)$$

3^o For k an odd integer and $L = (k+1)/2$, the frequency domain definition of the L-Wigner distribution is obtained:

$$MW_k(t,\omega) = \int_\theta P(\theta) X^L(\omega + \frac{\theta}{2L}) X^{*L}(\omega - \frac{\theta}{2L}) e^{-j\theta t} d\theta \qquad (37)$$

The last form is described in details in [5], [6], [17].

VII. CONCLUSION

Wigner higher order spectra (with a special attention to the Wigner bispectrum) are analyzed in the case of multicomponent signals. It is shown that the original definitions are useless in their basic forms. The modifications of spectra are proposed in order to treat this kind of signals. If the order of Wigner higher order spectra is an even one, then the additional signal transformation is needed. The theory is illustrated with a nonstationary multicomponent signal.

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