An Analysis of Wigner Higher Order Spectra of Multicomponent Signals

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Abstract— **The analysis of multicomponent signal representation by the Wigner higher order spectra is done. It is shown that the cross-terms can be removed only for an odd order, having equal number of conjugated and nonconjugated terms in the local multidimensional moment function. A method for higher order time-multifrequency analysis that eliminates cross-terms is proposed. This method turns out to be a dual definition of the L-Wigner distribution. The theory is illustrated by the numerical example.**

I. INTRODUCTION

Higher order spectral analysis has been intensively studied during last few years. Its applications are in many fields: radars, sonars, biomedicine, plasma physics, seismic data processing, image reconstruction, time-delay estimation, adaptive filtering...Higher order statistics, known as cumulants, and its Fourier transforms (FT) known as higher order spectra (polyspectra) are often considered, but we refer here only to the review paper, [1] and the references therein. Recently, higher order time-varying spectra have been defined and analyzed, [2]. The basic representation in time-varying higher order spectral analysis is the Wigner higher order spectra, like the Wigner distribution in the case of time-frequency analysis. The definition of Wigner higher order spectra of order k of a complex deterministic signal $x(t)$ is given, [2], by:

$$W_k(t,\omega_1,\omega_2,...,\omega_k)=\int_{\tau_1}\int_{\tau_2}...\int_{\tau_k}$$

$$R_{tk}(\tau_1,\tau_2,...,\tau_k)\prod_{i=1}^{k}(e^{-j\omega_i\tau_i}d\tau_i) \qquad (1)$$

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where R_{tk} is the local k-dimensional moment function, used in [2] as:

$$R_{tk}(\tau_1,\tau_2,...,\tau_k)=x^*(t-\alpha)\prod_{i=1}^{k}x(t-\alpha+\tau_i) \qquad (2)$$

with an arbitrary time delay α.

One of the substantial properties of a time-frequency distribution is that its mean frequency over the multifrequency space, defined as:

$$E\{\omega_m\}=$$

$$=\frac{\int_{\omega_1}...\int_{\omega_k}\omega_m W_k(t,\omega_1,\omega_2,...,\omega_k)\prod_{i=1}^{k}d\omega_i}{\int_{\omega_1}...\int_{\omega_k}W_k(t,\omega_1,\omega_2,...,\omega_k)\prod_{i=1}^{k}d\omega_i},$$

$$m=1,2,...,k$$

is equal to the scaled value of instantaneous frequency of signal $x(t)=R(t)exp(j\varphi(t))$, i.e. $E\{\omega_m\}=C\varphi\prime(t)$, for $m=1,2,...k$, [2],[7]. Assuming, as in [2], that α is a linear function of $\tau_i, \alpha=\sum_{i=1}^{k}a_i\tau_i$, it may be shown that the previous condition is satisfied for:

$$\alpha=\frac{1}{k+1}\sum_{i=1}^{k}\tau_i \qquad (3)$$

It means that the local moment function R_{tk} has to be centered at the time instant t:

$$\frac{1}{k+1}[(t-\alpha)+\sum_{i=1}^{k}(t+\tau_i-\alpha)]=t$$

The Wigner higher order spectra in terms of the Fourier transform $X(\omega)$ of a signal $x(t)$, may be easily derived from (1),(for the details of derivation see [2]):

$$W_k(t,\omega_1,\omega_2,...,\omega_k)=$$

$$= \frac{1}{2\pi} \int_{\theta} X^*(\sum_{i=1}^{k} \omega_i + \frac{\theta}{k+1})$$

$$\times \prod_{i=1}^{k} X(\omega_i - \frac{\theta}{k+1}) e^{-j\theta t} d\theta \qquad (4)$$

In the sections that follow we will analyze the auto-terms and cross-terms presence and locations in the Wigner higher order spectra (the case $k = 1$, i.e. the Wigner distribution, is analyzed in [4],[8],[9]). On the basis of this analysis, a method for cross-terms removal (or reduction) will be presented.

II. AUTO-TERMS AND CROSS-TERMS IN THE WIGNER HIGHER ORDER SPECTRA

A. Multicomponent Long Duration Signals

Consider a multicomponet signal $x(t)$ of the form:

$$x(t) = \sum_{m=1}^{M} R_m(t) e^{j\varphi_m(t)} \qquad (5)$$

where the instantaneous frequencies $\varphi'_m(t)$ and amplitudes $R_m(t)$ are such that:

1) The Fourier transform of $R_m(t)$ is band-limited $FT\{R_m(t)\} = \mathbf{R}_m(\omega) = 0$ for $|\omega| > W_R$, with a small W_R comparing to the considered frequency interval in the time-frequency domain;

2) $\mathrm{FT}\{R_m(t)e^{j\varphi(t)}\} = \mathbf{R}_m(\omega - \varphi'_m(t))$, meaning that the instantaneous frequency may be treated as a constant inside the considered time interval (or inside the window) in the time-frequency domain.

Here, we show that the cross-terms in the Wigner higher order spectra, defined by (1) and (2), can not be removed. However, the cross-terms removal is possible if we follow the idea of equal number of conjugate and nonconjugate terms in (2). That option was preferred in [1] in the cumulant analysis. This idea may be exploited only if the order k is an odd one, i.e. $k = 2r - 1$.

In order to prove the above statement consider a general form of the Wigner higher order spectra defined as a k-dimensional Fourier transform of the local moment function:

$$R_{tk}(\tau_1, \tau_2, ..., \tau_k) =$$

$$= x^*(t-\alpha) \prod_{i=1}^{l-1} x^*(t-\alpha+\tau_i) \prod_{i=l}^{k} x(t-\alpha+\tau_i)$$

with

$$W_k(t, \omega_1, \omega_2, ..., \omega_k) = FT_\tau\{R_{tk}(\tau_1, \tau_2, ..., \tau_k)\} \qquad (6)$$

where l is an integer, $0 < l \leq k$. Note that, for $l = 1$, relation (6) reduces to eqn.(2).

Again, the mean conditional frequency is at the equal distance from the origin, along the frequency axes $\omega_1, \omega_2, ..., \omega_k$ if α is the same as in (2), i.e. if we preserve the property that R_{tk} is centered at the time instant t.

Substituting the signal x(t) in terms of its Fourier transform:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

in (6) with α defined by (3), we get, after some straightforward transformations, that the frequency domain definition of the Wigner higher order spectra defined by (6), is:

$$W_k(t, \omega_1, \omega_2, ..., \omega_k) =$$

$$= \frac{1}{2\pi} \int_{\theta} X^*(\sum_{i=1}^{k} \omega_i + \frac{\theta}{k+1})$$

$$\times \prod_{i=1}^{l-1} X^*(-\omega_i + \frac{\theta}{k+1}) \prod_{i=l}^{k} X(\omega_i - \frac{\theta}{k+1}) e^{-j\theta t} d\theta \qquad (7)$$

Let us consider the location of auto-terms in (7). Obviously, taking $M = 1$, the cross-terms do not exist. The integrand in (7) is different from zero only if the following inequalities hold:

$$\left| \frac{\theta}{k+1} - \frac{(2l-1-k)\varphi'(t)}{k+1} \right| < W_R$$

$$\left| -\omega_m + \frac{(2l-1-k)\varphi'(t)}{k+1} - \varphi'(t) \right| < 2\varepsilon W_R$$

$$\left| \omega_n - \frac{(2l-1-k)\varphi'(t)}{k+1} - \varphi'(t) \right| < 2\varepsilon W_R$$

$$m = 1, 2, ...l-1 \qquad n = l, l+1, ..., k$$

$$\text{with } \varphi'(t) \equiv \varphi'_1(t),\ \varepsilon = k/(k+1) \qquad (8)$$

The location of auto-term along θ axis depends on the signal's instantaneous frequency for any l, except for the case $l = r = (k+1)/2$, when the values of integrand are concentrated around the θ-axis origin. In that case the auto-terms are centered along the line s in the k-dimensional ω space, defined by:

$$s \quad : \quad \omega_1 = -\omega, \omega_2 = -\omega, ..., \omega_{r-1} = -\omega$$
$$\omega_r \quad = \quad \omega, \omega_{r+1} = \omega, ..., \omega_k = \omega$$

at the point $\omega = \varphi'(t)$.

This line is a symmetral of the k-dimensional quadrant in which the negative values of $\omega_1, \omega_2, ..., \omega_{r-1}$, and the positive values of $\omega_r, \omega_{r+1}, ..., \omega_k$ lie.

The illustration of Wigner bispectrum (eqns. (1) and (2) with $k = 2$)[1] which does not satisfy the condition for cross-terms removal, as well as the illustration of Wigner trispectrum (eqns. (1) and (6) with $k = 3$), as the lowest one satisfying the previous conditions (excluding the well-known Wigner distribution), are given in Figs.1. and 2. respectively.

If we take $M > 1$, then the integrand in (7) is different from zero, along the line s, for:

$$\left|\frac{\theta}{k+1} - \frac{\varphi_i'(t) - \varphi_j'(t)}{k+1}\right| < W_R$$
$$i, j = 1, 2, ...M$$
$$\left|-\omega_m + \frac{\varphi_i'(t) - (k+2)\varphi_j'(t)}{k+1}\right| < 2W_R$$
$$m = 1, 2, ..., r-1$$
$$\left|\omega n - \frac{\varphi_i'(t) + k\varphi_j'(t)}{k+1}\right| < 2W_R$$
$$n = r, r+1, ..., 2r-1 \qquad (9)$$

From (8) and (9) we can draw two essential conclusions for the derivation of a method for cross-terms removal: i) If $k = 2l - 1$, the values of integrand, corresponding to the auto-terms $(i = j)$, are different from zero only around the origin $(\theta = 0)$, for any instantaneous frequency; ii) The cross-terms are dislocated from the origin of θ axis.

Introducing the frequency domain window $P(\theta)$ of the width $2W_m$ i.e. $P(\theta) = 0$ for $\theta > W_m$, the Wigner higher order spectra (given by (7)) along the defined line s become:

$$W_k(t, \omega) = \frac{1}{2\pi}\int_\theta X^{*r}(\omega + \frac{\theta}{2r})$$
$$\times X^r(\omega - \frac{\theta}{2r})e^{-j\theta t}P(\theta)d\theta \qquad (10)$$

The distribution defined by (10) may be easily directly implemented in the frequency domain using the Fourier transform $X(\omega)$ of the signal $x(t)$ and the frequency domain window $P(\theta)$. An alternative and very efficient approach to the calculation of the distributions of form (10) is described in [4].

From (9) one may conclude that the integration over auto-terms is completely performed and, at the same time, the cross-terms are removed if the width $2W_m$ of $P(\theta)$ satisfies:

$$(k+1)W_R < W_m <$$
$$< min_{i,j}|\varphi_i'(t) - \varphi_j'(t)| - (k+1)W_R \qquad (11)$$

The relation (11) tells that the width of window $P(\theta)$ must be wider than the maximum bandwidth of the signal component amplitudes $R_m(t), m = 1, 2, ..., k$, multiplied by $(k+1)$ and, at the same time, narrower than the minimal distance between the instantaneous frequencies of any two components $R_i(t)exp(j\varphi_i(t))$ and $R_j(t)exp(j\varphi_j(t))$ of the signal $x(t)$ decreased for $(k+1)W_R$. If the first condition is not satisfied the distortion of original Wigner higher order spectra will appear and if the second condition is not satisfied the cross-terms will appear.

This way, the distribution described by (10) may preserve all appealing properties of the Wigner higher order spectra (described in [2]), while, at the same time, avoiding the presence of cross-terms[2].

[1]The definition of the Wigner bispectrum is:

$$W_2(t, \omega_1, \omega_2) =$$
$$\int_{\tau_1}\int_{\tau_2} x^*(t - \tau_1/3 - \tau_3/3)x(t + 2\tau_1/3 - \tau_2/3)$$
$$\times x(t - \tau_1/3 + 2\tau_2/3)e^{-j\omega_1\tau_1 - j\omega_2\tau_2}d\tau_1 d\tau_2$$

[2]Note thet the Wigner distribution is a special case with $l = k = 1$. An analysis of cross terms in the Wigner distribution is done in [4].

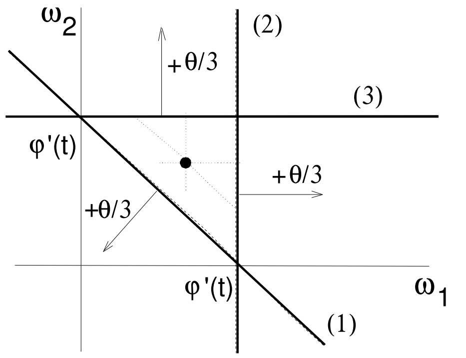

Fig. 1. Illustration of Wigner bispectrum $(1): \omega_1+\omega_2+\theta/3=\varphi\prime(t)$; $(2): \omega_1-\theta/3=\varphi\prime(t)$; $(3): \omega_2-\theta/3=\varphi\prime(t)$.

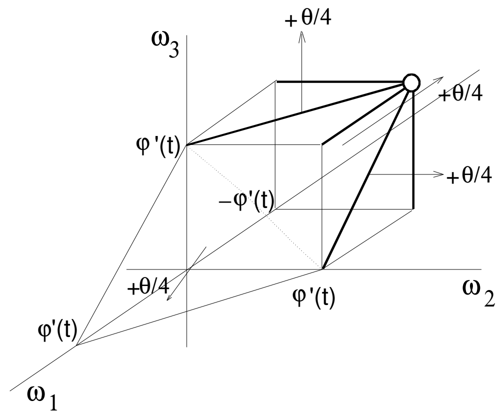

Fig. 2. Illustration of Wigner trispectrum

Note that the definition (10) represents just a dual definition of the L-Wigner distribution in time domain:

$$LWD(t,\omega)=\int_\tau x^{*L}(t-\frac{\tau}{2L})$$

$$\times x^L(t+\frac{\tau}{2L})e^{-j\omega\tau}w(\tau)d\tau \qquad (12)$$

We proposed the L-Wigner distribution, [5],[6],[10], in order to linearize the instantaneous frequency when it is not a linear function of time. The properties of the L-Wigner distribution have been examined in details in the above references. The same computational techniques and other results may be applied to the Wigner higher order spectra.

B. Multicomponent Short Duration Signals

Consider a multicomponet signal formed as a sum of signals:

$$x(t)=\sum_{m=1}^{M}x_m(t-t_m) \qquad (13)$$

where $x_m(t)$ are such that $x_m(t)=0$ for $|t|>T_p$, with T_p small comparing to the considered time interval in the time-frequency domain.

From (1) and (6) it is easy to see that all auto-terms are located around the origin of τ coordinate system, i.e. around $\tau_i=0$ $(i=1,2,...k)$. The cross-terms are dislocated from the origin. They lie around $\tau_i=t_m-t_n$, so they may be eliminated using a window $w(\tau_1,\tau_2...,\tau_k)=0$ for $|\tau_i|>T$, where:

$$T_p<T<min_{n,m}|t_n-t_m|-T_p=T_m \qquad (14)$$

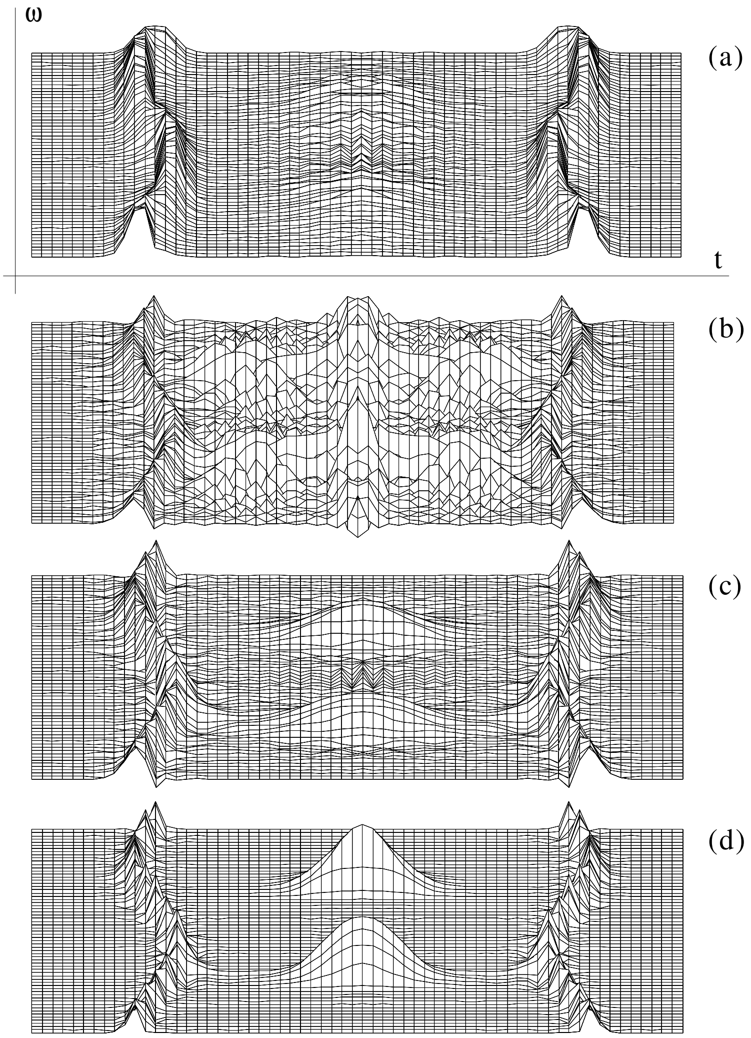

Fig. 3. Time-frequency representation of a multicomponent signal plus noise. a) Spectrogram; b) Wigner distribution; c) Wigner distribution without cross-terms (SM); d) Wigner trispectrum along line s without cross-terms; with: the rectangular window $w(\tau_1, \tau_2, \tau_3)$ and the Hanning window $P(\theta)$; SNR=15 dB; considered intervals $-10 < t < 10$; $-10 < \omega < 10$.

When we deal with a sum of two multicomponent signals, one of form (5) and the other of form (13), we may use both windows $P(\theta)$ and $w(\tau_1, \tau_2 ..., \tau_k)$. In that case the Wigner higher order spectra, along the line s, are:

$$W_k(t,\omega) = p(t) \otimes_t$$

$$\int_{\tau_1}\int_{\tau_2}...\int_{\tau_k} w(\tau_1,\tau_2,...,\tau_k) R_{kt}(\tau_1,\tau_2,...,\tau_k)$$

$$\times e^{-j\omega(-\tau_1-\tau_2-...-\tau_{r-1}+\tau_r+...+\tau_k)} d\tau_1 d\tau_2 ... d\tau_k$$

$$R_{kt}(\tau_1,\tau_2,...,\tau_k) =$$

$$x^*(t-\alpha)\prod_{i=1}^{r-1} x^*(t-\alpha+\tau_i)\prod_{i=r}^{k} x(t-\alpha+\tau_i)$$

with $\otimes_t$ formally denoting convolution in time domain, but basically the product in frequency domain as it is applied in this method, eqn.(10).

The presence of cross-terms is dependent on the shape of windows $w(\tau_1, \tau_2, ..., \tau_k)$ and $p(t)$. The window $p(t)$ (i.e. its frequency domain version$P(\theta)$, eqns.(10), (11)) controls the cross-terms along the frequency axis and the window $w(\tau_1, \tau_2, ..., \tau_k)$, eqn.(14), controls the cross-terms along the time-axis in the time-frequency domain.

III. NUMERICAL EXAMPLE

In the numerical example we consider the multicomponent signal:

$$x(t) = x_1(t) + x_2(t) + n(t),$$

$$x_1(t) = \frac{1.5}{\sqrt{2\pi}} e^{-\frac{t^2}{8}} \cos(4.5t)$$

$$x_2(t) = FT^{-1}\{\cos[2\cos((\omega + 10)/3)$$

$$-2.2\pi\omega] e^{-\frac{\omega^2}{320\pi}}\}$$

The components of $x_1(t)$ are of form (5) and the components of $x_2(t)$ are of form (13). The last component, $n(t)$, is a Gaussian white noise. The spectrogram of $x(t)$ is given in Fig.3a. The Wigner distribution of $x(t)$, with and without cross-terms, is shown in Figs.3b. and 3c, respectively. The Wigner trispectrum without cross-terms (SM), calculated using (10), is given in Fig.3d. Considered frequency interval is $-10 < \omega \leq 10$. The energies of signals $x_1(t)$ and $x_2(t)$ and noise $n(t)$ are: $E_1 = 0.635$, $E_2 = 1.5$ and $N = 0.07$, respectively in the considered frequency interval. The total SNR is 15dB. The advantages of trispectrum, as well as the efficiency of the proposed method (described by (10)) in cross-terms removal, are apparent from Fig.3.

Even the lowest of Wigner higher order spectra, the Wigner distribution, is almost useless without windowing because of very exhibited cross-terms, Fig. 3b. That effect is even more emphasized as the order k raises. But, using the proposed windows, $P(\theta)$ or/and $w(\tau_1, \tau_2, ..., \tau_k)$, whose characteristics are described, the cross-terms are removed keeping the appealing properties, [2], of Wigner higher order spectra. That may be observed in Figs. 3c,3d.

Besides, using the definition proposed in this paper (eqn.(10)) all those properties may be exploited from the Wigner higher order spectra on the single line s. That reduces the origially $k + 1$ dimensional Wigner higher order spectra to the two-dimensional one and makes its application easier.

IV. CONCLUSION

Wigner higher order spectra of multicomponent signals are analyzed. It is shown that the cross-terms may be removed only when the Wigner spectrum is of an odd order. A method for cross-terms removal is presented. The theory is illustrated on the numerical example.

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