Fast communication

Time–frequency representation based on the reassigned S-method

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Abstract

The paper presents a reassignment based method for improvement of the time–frequency representations. The S-method is used as a basis for the reassignment. Very simple reassigned form is proposed and illustrated on the examples. © 1999 Published by Elsevier Science B.V. All rights reserved.

1. Introduction and review

A method for improved distribution concentration, based on the reassignment of distribution values in the time–frequency plane has been proposed by Kodera et al. [9]. It has been reintroduced for the readability improvement of time–frequency and time–scale distributions by Auger and Flandrin [1,4]. This method is known as the reassignment method, and it can be of help in parametric signal identification [2].

This paper extends the reassignment approach to the S-method [12]. The S-method produces time–frequency representation close to the pseudo-Wigner distribution, avoiding cross-terms [12–14]. The reassigned S-method turns out to be very efficient and numerically less consuming than the other reassigned distribution forms. A simplified form of the reassignment for the S-method is proposed. Examples, including noisy signals, are given as the illustration of the presented procedure.

The reassigned form of a distribution from the Cohen class [1,5,8]

$$\text{CD}(t,\omega)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Pi(u,v) \text{WD}(t-u, \omega-v)\, \mathrm{d}u\, \mathrm{d}v \quad (1)$$

is given by

$$\text{RCD}(t,\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{CD}(t',\omega')\delta(t - t_r(t',\omega'))$$

$$\times \delta(\omega - \omega_r(t',\omega'))\, \mathrm{d}t'\, \mathrm{d}\omega',$$

$$(2)$$

where $\text{WD}(t,\omega)$ is the Wigner distribution, while $\Pi(t,\omega)$ is a distribution kernel in time–frequency domain. The reassignment method calculation may be understood as assigning the values of a distribution to a center of gravity of the considered region,

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where

$t_r(t', \omega')$

$$= t' - \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u\Pi(u,v)\mathrm{WD}(t'-u, \omega'-v)\,\mathrm{d}u\,\mathrm{d}v}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Pi(u,v)\mathrm{WD}(t'-u, \omega'-v)\,\mathrm{d}u\,\mathrm{d}v}$$

(3)

and

$\omega_r(t', \omega')$

$$= \omega' - \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v\Pi(u,v)\mathrm{WD}(t'-u, \omega'-v)\,\mathrm{d}u\,\mathrm{d}v}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Pi(u,v)\mathrm{WD}(t'-u, \omega'-v)\,\mathrm{d}u\,\mathrm{d}v}.$$

(4)

It is interesting to note that for the Wigner distribution, when $\Pi(t, \omega) = 2\pi\delta(t)\delta(\omega)$, the reassigned distribution is identical to the original distribution since in that case $t_r(t', \omega') = t'$ and $\omega_r(t', \omega') = \omega'$.

The reassigned form of the spectrogram is obtained using general expressions (3), (4), as [1]

$$t_r(t', \omega') = t' + \mathrm{Re}\left\{\frac{\mathrm{STFT}_{\tau_w}(t', \omega')}{\mathrm{STFT}_w(t', \omega')}\right\},$$

$$\omega_r(t', \omega') = \omega' - \mathrm{Im}\left\{\frac{\mathrm{STFT}_{D_w}(t', \omega')}{\mathrm{STFT}_w(t', \omega')}\right\},$$

(5)

where $\tau_w = \tau w(\tau)$, $D_w = \partial w(\tau)/\partial \tau$ and

$$\mathrm{STFT}_w(t, \omega) = \int_{-\infty}^{\infty} x(t+\tau)w^*(\tau)\mathrm{e}^{-\mathrm{j}\omega\tau}\,\mathrm{d}\tau.$$

Examples of the reassigned forms of some other reduced interference distributions may be found in [1,11].

2. Reassigned S-method

The S-method is defined as [12–14]

$\mathrm{SM}_{w,w}(t, \omega)$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} P(\theta)\mathrm{STFT}_w(t, \omega+\theta)\mathrm{STFT}_w^*(t, \omega-\theta)\,\mathrm{d}\theta.$$

It belongs to the Cohen class of distributions. Kernel of the S-method in time–frequency domain is given by $\Pi(t, \omega) = 2p(2t)\mathrm{WD}_w(t, \omega)$, where $p(t) = \mathrm{IFT}\{P(\theta)\}$, and $\mathrm{WD}_w(t, \omega)$ is the Wigner distribution of a lag-window function.

For a multicomponent signal, with an appropriate window $P(\theta)$ width, the S-method may produce a distribution close to the sum of the pseudo-Wigner distributions of each individual signal component, $\mathrm{SM}_{w,w}(t, \omega) \approx \sum_{i=1}^{M} \mathrm{WD}_{x_ix_i}(t, \omega)$. This interesting property has already turned the attention of numerous other researchers to the S-method [3,6,7].

Readability of the time–frequency representation using the S-method may be improved by reassigning its values according to

$$t_r(t, \omega) = t + \mathrm{Re}\left\{\frac{\mathrm{SM}_{\tau_w,w}(t, \omega)}{\mathrm{SM}_{w,w}(t, \omega)}\right\},$$

(6)

$$\omega_r(t, \omega) = \omega - \mathrm{Im}\left\{\frac{\mathrm{SM}_{D_w,w}(t, \omega)}{\mathrm{SM}_{w,w}(t, \omega)}\right\},$$

where the indexes in $\mathrm{SM}_{v,w}(t, \omega)$ denote the windows used in the corresponding STFT calculation.

The expression for $t_r(t', \omega')$ can significantly be simplified for the rectangular window $P(\theta)$, as it is used in the S-method. Then[1]

$t_r(t, \omega)$

$$= t + \mathrm{Im}\left\{\frac{\mathrm{STFT}_w(t, \omega+\theta_P)\mathrm{STFT}_w^*(t, \omega-\theta_P)}{\mathrm{SM}_{w,w}(t, \omega)}\right\},$$

(7)

where $2\theta_P$ is the window $P(\theta)$ width, $P(\theta) = 0$ for $|\theta| > \theta_P$. As expected, if the window $P(\theta)$ is wider than the auto-term width (width of the corresponding $|\mathrm{STFT}_w(t, \omega)|$), then for that point $\mathrm{STFT}_w(t, \omega+\theta_P)\mathrm{STFT}_w^*(t, \omega-\theta_P) \cong 0$, thus $t_r(t, \omega) = t$. This is the same as in the Wigner distribution case, since the S-method and the Wigner distribution are equal for that auto-term. *It proves once more the fact that, in this case, the S-method is locally equal to the Wigner distribution.*

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[1] If the frequency domain window $P(\theta)$ is rectangular, with the width $2\theta_p$, then [10]

$$\int_{-\infty}^{\infty} G(\omega, t, \theta)\frac{\mathrm{d}P(\theta)}{\mathrm{d}\theta}\,\mathrm{d}\theta = G(\omega, t, -\theta_p) - G(\omega, t, \theta_p).$$

Discrete form of the reassignment displacement is

$$n_r(n, k)$$

$$= n + \mathrm{Im}\left\{\frac{\mathrm{STFT}_w(n, k + L)\mathrm{STFT}_w^*(n, k - L)}{|\mathrm{STFT}_w(n, k)|^2 + M(n, k)}\right\},$$

$$\tag{8}$$

with

$$M(n, k)$$

$$= 2\,\mathrm{Re}\sum_{i=1}^{L}\mathrm{STFT}_w(n, k + i)\mathrm{STFT}_w^*(n, k - i).$$

This is a very simple form for the applications. As it will be shown by examples, it may significantly improve the results, approaching the Wigner distribution without cross-terms case. Relation (8) requires the STFT calculation *using only one window*. Thus, it can be numerically very efficient.

Note that the realization of $t_r(n, k)$ can be done in a form appropriate for the VLSI implementations, using the recursive STFT(n, k) relation [10,11,14]

$$\mathrm{STFT}_R(n, k) = [\mathrm{STFT}_R(n - 1, k)$$

$$- x(n - N/2) + x(n + N/2)]\mathrm{e}^{\mathrm{j}(2\pi/N)k},$$

$$\tag{9}$$

where N is the rectangular window width. The modification for the other window types, like for example for the Hanning window, is

$$\mathrm{STFT}_H(n, k)$$

$$= \tfrac{1}{2}\mathrm{STFT}_R(n, k)$$

$$+ \tfrac{1}{4}[\mathrm{STFT}_R(n, k - 1) + \mathrm{STFT}_R(n, k + 1)].$$

$$\tag{10}$$

Also, we can calculate the STFT$_{D_w}$ based on the STFT with rectangular window. For example if $w(t)$ is the Hanning window then

$$\mathrm{STFT}_{D_w}(n, k)$$

$$= \frac{\mathrm{j}\pi}{2NT}[\mathrm{STFT}_R(n, k - 1) - \mathrm{STFT}_R(n, k + 1)],$$

where T is the sampling interval.

3. Simplified form

Further simplification may be achieved if we use the reassignment along the time-axis only. Since the S-method is already close to the Wigner distribution case, the values of time and frequency displacement are small, converging to zero. Thus, the reassignment along only one direction, time or frequency, can produce good results. Because of the implementation simplicity, according to (8), we will use time reassignment only

$$\mathrm{RSM}(t, \omega)$$

$$= \int_{-\infty}^{\infty}\mathrm{SM}_{w,w}(t', \omega')\delta(t - t_r(t', \omega'))\,\mathrm{d}t',$$

$$\tag{11}$$

where $t_r(t', \omega')$ is given by (7), (8).

Note that the S-method does not require either the signal oversampling with factor of two or the analytic signal calculation for the real-valued signals. These are sources of additional computational savings with respect to the common approach of the reassigned smoothed Wigner distribution, where the Wigner distribution is calculated and then smoothed, before the reassignment method is applied.

4. Examples

Example 1. Signal of the form

$$x(t) = x_1(t) + x_2(t),$$

with

$$x_1(t) = \mathrm{e}^{\mathrm{j}80\pi t} + \mathrm{e}^{-\mathrm{j}9\pi((t+1.75)^3 - 7.75t)},$$

$$-3 \leqslant t \leqslant -\tfrac{1}{4},$$

$$x_2(t) = \mathrm{e}^{-\mathrm{j}8\pi t^2 + \mathrm{j}81\pi t} + \mathrm{e}^{-8(t-2.5)^2}\mathrm{e}^{-\mathrm{j}8\pi t^2 + \mathrm{j}109\pi t},$$

$$0 \leqslant t \leqslant 3,$$

is considered. The sampling interval is $T = \tfrac{1}{96}$. The spectrograms of this signal, calculated using a wide ($N = 192$) and a narrow ($N = 32$) Hanning window, are shown in Fig. 1(a,b). The S-method with $L = 2$, based on the STFT with a wide window $N = 192$, is shown in Fig. 1(c). It is close to the sum
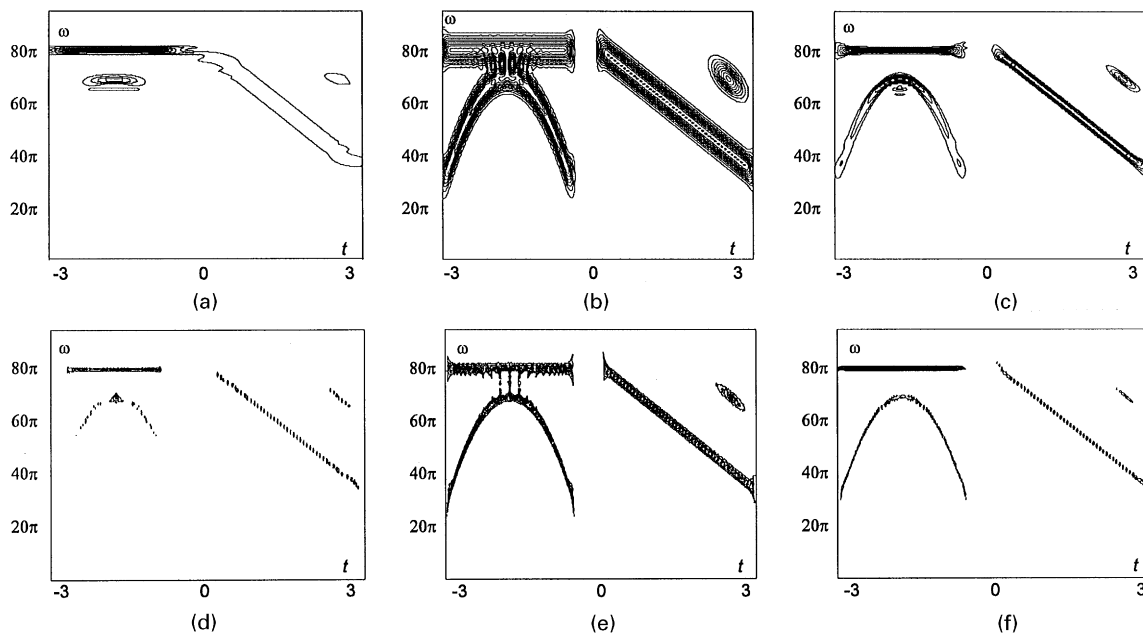
Fig. 1. Time–frequency representation of a multicomponent signal: (a) spectrogram with a wide lag-window, (b) spectrogram with a narrow lag-window, (c) S-method based on the STFT with a wide lag-window, (d) reassigned spectrogram form (a), (e) reassigned spectrogram form (b), reassigned S-method.

of the Wigner distributions of each signal component individually. In the numerical realizations of the reassignment-based methods a threshold should be assumed [1]. All distribution values below the threshold are then neglected in the reassignment calculations. The threshold is here assumed at 5% of the corresponding distribution maximal value. The reassigned version of spectrogram with wide window cannot produce complete concentration along the instantaneous frequency Fig. 1(d). The reassigned spectrogram with narrow window does not separate close signals component Fig. 1(e). The reassigned S-method produces almost complete concentration along the instantaneous frequency Fig. 1(f). The S-method with a time reassignment only, produces very similar results as the S-method with reassignment in both directions (Fig. 2). The last representation is obtained in a numerically very efficient way.

Example 2. An interesting application of time–frequency distributions is in parametric identification
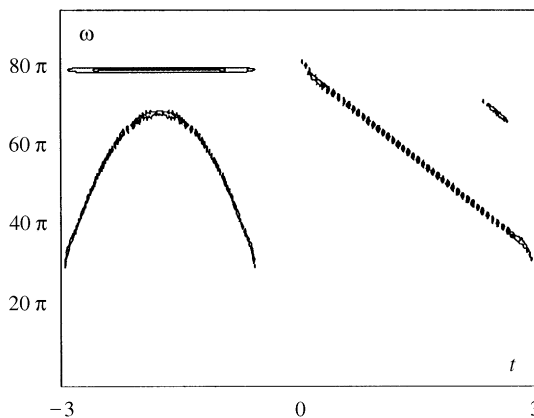

Fig. 2. Time–frequency representation of a multicomponent signal based on the time direction reassigned S-method.

of signals corrupted with a noise [2]. For this purpose it is very important that a time–frequency distribution highly concentrates signals' energy along the instantaneous frequency. Reassignment
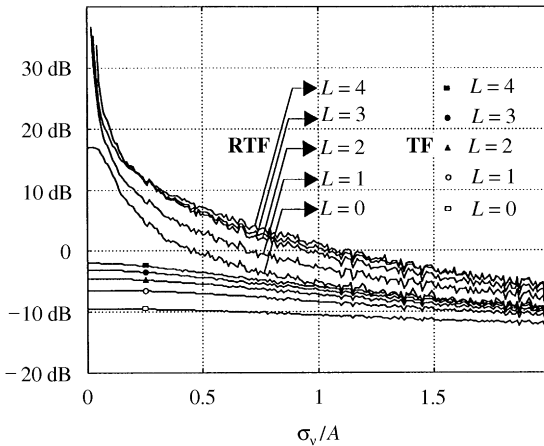
Fig. 3. The ratio of distribution concentration along the instantaneous frequencies and distribution concentration outside these regions for different distributions, as a function of σ_v/A. Reassigned forms are denoted by RTF.

distributions perfectly localized the chirps. So consider a sum of two linear frequency modulated signals

$$x(t) = e^{j16\pi t^2 + j24\pi t} + e^{j16\pi t^2 - j24\pi t}$$

corrupted with the white Gaussian noise of variance σ_v^2. The spectrogram ($L = 0$), the S-method with $L = 1, 2, 3, 4$, and their corresponding reassigned forms (RTF) are considered. The ratio of the energy along the instantaneous frequencies and the energy outside these regions is defined as

$$B = 10 \log \frac{\iint_{(t,\omega) \in \mathbf{R}} \mathrm{TF}(t, \omega) \, \mathrm{d}t \, \mathrm{d}\omega}{\iint_{(t,\omega) \notin \mathbf{R}} \mathrm{TF}(t, \omega) \, \mathrm{d}t \mathrm{d}\omega}.$$

Region R corresponds to the instantaneous frequency lines of the signal components. This ratio for different distributions (different L), as a function of σ_v/A, is shown in Fig. 3. The nonreassigned distributions are denoted by TF. For example, for $\sigma_v/A = 1$ the reassigned S-method for $L = 4$ gives B for 6.5 dB greater than in the reassigned spectrogram ($L = 0$) and nonreassigned S-method cases. For frequency window $P(\theta)$ wider than the autoterms, the S-method cannot produce further auto-terms concentration improvement while

it can comprise noise that degrades considered ratio [13].

5. Conclusion

The reassigned form of the S-method is proposed. It is qualitatively and computationally very efficient. A simplified form, based on the reassignment along time axis only, is presented. The method efficiency is illustrated on examples.

Acknowledgements

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