Unified Approach to the Noise Analysis in the Spectrogram and Wigner Distribution

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Abstract— **An analysis of time-frequency representations of noisy signals is perfomed. Using the S-method for time-frequency signal analysis, the influence of noise on two the most important distributions (spectrogram and Wigner distribution) is analyzed in unified manner. It is also shown that, for signals whose instantaneous frequency is not constant, an improvement over the spectrogram and the Wigner distribution performances in a noisy environment may be achieved using the S-method. The expressions for mean and variance are derived. Results are given for several illustrative and numerical examples.**

I. INTRODUCTION

Time-frequency signal representations may roughly be classified into two categories: linear and quadratic [1], [2] (recently, higher order representations have been introduced, [3], [4], [5]). From the linear class of transforms, we will mention only the most important one, the short-time Fourier transform (STFT). The second class of time-frequency transforms are quadratic one. Despite the absence of the linearity property, they are frequently used because many aspects of the signal's representation may be improved with respect to the linear transforms. The Wigner distribution (along with its pseudo and smoothed forms) is the most prominent member of this class. It satisfies most of the desired properties of a time-frequency distribution [1], [2], [10], [22]. This is the reason for its wide applicability and research interest. Besides the Wigner distribution, there are other important quadratic distributions. It is important to note that all quadratic (shift covariant) time-frequency distributions belong to the Cohen class [1].

The energetic version of the STFT, called a spectrogram, also belongs to this class. The Wigner distribution, in contrast to the spectrogram and some other distributions (reduced interference distributions [1], [2], [6], [7], [8], [9], [22]), exhibits very strong cross-term effects when multicomponent signals are analyzed.

The S-method, recently defined in [11] and analyzed in detail in [4], [5], [12], [13], [14], [23], is able to produce the representation of a multicomponent signal such that the distribution of each component is its Wigner distribution, but without cross-terms. The S-method may be implemented in a numerically very efficient way (more efficient than that of the Wigner distribution itself) [11]. Two special (marginal) cases of the S- method, which follow, are just two the most frequently used distributions: the spectrogram and the Wigner distribution.

Although noise is very often present in the considered time-frequency signal representations, its rigorous analytical treatment has been dealt with only in a few papers. Martin and Flandrin analyzed time-frequency representations of nonstationary random processes in [15]. Nuttal analyzed noisy signals in [16], while the analog and discrete forms of the Wigner distribution of noisy signals were studied by Stanković and Stanković in [17], [18]. Hearon and Amin considered the variance in the Cohen class of distributions and found the optimal kernel with respect to noise variance, [19].

In this paper, noise is analysed in the framework of the S-method. It is shown that, under some conditions, the application of the S-method in the case of noisy signals may im-

prove the performances with respect to the spectrogram and Wigner distribution. This paper is organized as follows. In the next section, a short review of the S-method is given. Analysis of the time-frequency representation of noisy signals is presented in Section III. Examples are provided in Section IV.

II. REVIEW OF THE S-METHOD

Let us consider two basic means for the time-frequency analysis of a signal: the short-time Fourier transform (STFT) and the Wigner distribution (WD), [1], [2], [10]. The STFT and the pseudo-form of the WD are defined by:

$$STFT_f(t,\omega) = \int_{-\infty}^{\infty} f(t+\tau)w(\tau)e^{-j\omega\tau}d\tau \quad (1)$$

$$PWD_f(t,\omega) = \int_{-\infty}^{\infty} w\left(\frac{\tau}{2}\right)w\left(-\frac{\tau}{2}\right) \times$$
$$f\left(t+\frac{\tau}{2}\right)f^*\left(t-\frac{\tau}{2}\right)e^{-j\omega\tau}d\tau, \quad (2)$$

where the real window $w(\tau)$ is assumed. Relation between the STFT and the PWD is derived in [11] as:

$$PWD_f(t,\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} STFT_f(t,\omega+\theta)\times$$
$$STFT_f^*(t,\omega-\theta)d\theta. \quad (3)$$

On the basis of the previous expressions, the S-method for time-frequency analysis is given in the following form:

$$SM_f(t,\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} P(\theta)STFT_f(t,\omega+\theta)\times$$
$$STFT_f^*(t,\omega-\theta)d\theta. \quad (4)$$

Through a suitable selection of the window $P(\theta)$, it is possible to obtain the auto-terms of multicomponent signals such that they remain unchanged with respect to those in the WD, while the entire elimination (or reduction) of cross-terms is achieved, as shown in [5], [11], [12], [13].

The discrete forms of equations (1), (2) and (3) are [11], [12], [13]:

$$DSTFT_f(n,k) = \sum_{i=0}^{N-1} f(n+i)w(i)e^{-j\frac{2\pi}{N}ik},$$
$$\quad (5)$$

$$DPWD_f(n,k) = \sum_{i=0}^{N-1} w(i)w(-i)\times$$
$$f(n+i)f^*(n-i)e^{-j\frac{2\pi}{N}2ik}, \quad (6)$$

$$DPWD_f(n,k) = \frac{1}{N}\sum_{l=-\frac{N}{2}}^{\frac{N}{2}-1} DSTFT_f(n,k+l)\times$$
$$DSTFT_f^*(n,k-l). \quad (7)$$

Factor of 2 is omitted in (6) in order to simplify the notation. Relation (7) may be written in symmetrical form as:

$$DPWD_f(n,k) = \frac{1}{N}\sum_{l=-N/2}^{N/2} \alpha(l)\times$$

$$DSTFT_f(n,k+l)DSTFT_f^*(n,k-l), \quad (8)$$

where $\alpha(l) = 1$, for all l except $|l| = N/2$, when $\alpha(\pm N/2) = 1/2$. Understanding equation (8) as an averaged value of the STFT of a discrete signal and its complex conjugate value, the discrete form of the S-method is obtained as:

$$DSM_f(n,k) = \frac{1}{2L_d+1}\sum_{l=-L_d}^{L_d} P_d(l)\times$$

$$DSTFT_f(n,k+l)DSTFT_f^*(n,k-l). \quad (9)$$

Note that:

1) for $P_d(l) = \delta(l)$, we obtain the spectrogram of discrete signals (DSPEC), and

2) for $P_d(l) = 1$ and $2L_d + 1 = N$, the discrete form of the Wigner distribution (in this case $\alpha(l)$ should be included).

Taking into consideration that:

$$DSTFT(n,k+l)DSTFT^*(n,k-l)+$$

$$DSTFT(n,k-l)DSTFT^*(n,k+l) =$$

$$2\text{Re}\left\{DSTFT(n,k+l)DSTFT^*(n,k-l)\right\},$$

and assuming that $P_d(l)$ is a rectangular window, we get:

$$DSM_f(n,k) = DSPEC_f(n,k)+$$

$$\frac{2}{2L_d+1}\sum_{l=1}^{L_d} \text{Re}\{DSTFT_f(n,k+l)\times$$

$$DSTFT_f^*(n, k - l)\}, \qquad (10)$$

where $DSPEC_f(n,k) = |DSTFT_f(n,k)|^2$.

Details on the numerical and on-line implementation of the distribution defined by (10), as well as its calculational complexity, are given in [5], [11], [12], [13], [23].

III. NOISY SIGNAL ANALYSIS

In this section, we assume that a deterministic signal $f(n)$ is corrupted by an additive noise $\nu(n)$, so that the time-frequency analysis will be performed on the basis of $x(n) = f(n) + \nu(n)$. Complex and real noise $\nu(n)$ will be considered. Very simple expressions for the mean and variance of the S-method (including the spectrogram and Wigner distribution as special cases) will be derived for Gaussian white noise.

A. Complex noise

Consider signal $f(n)$ with additive complex noise $\nu(n)$ with independent real and imaginary parts, having equal variances, denoted by $\sigma_\nu^2/2$. The total noise variance is σ_ν^2. In order to analyze the noise's influence on the S-method, we rewrite equation (9) in the form:

$$DSM_x(n,k) = \frac{1}{2L_d + 1} \times$$

$$\sum_{l=-L_d}^{L_d} \sum_{i_1=0}^{N-1} \sum_{i_2=0}^{N-1} w(i_1)w(i_2)x(n+i_1)x^*(n+i_2) \times$$

$$e^{-j\frac{2\pi}{N}k(i_1-i_2)}e^{-j\frac{2\pi}{N}l(i_1+i_2)}. \qquad (11)$$

A.1 Mean value of the S-method

The mean of $DSM_f(n,k)$ estimator, based on the discrete signal $x(n) = f(n) + \nu(n)$, is:

$$E\{DSM_x(n,k)\} = DSM_f(n,k) +$$

$$\frac{1}{2L_d + 1} \sum_{l=-L_d}^{L_d} \sum_{i_1=0}^{N-1} \sum_{i_2=0}^{N-1} w(i_1)w(i_2) \times$$

$$R_{\nu\nu}(i_1 - i_2)e^{-j\frac{2\pi}{N}k(i_1-i_2)}e^{-j\frac{2\pi}{N}l(i_1+i_2)}, \quad (12)$$

where $R_{\nu\nu}(i_1 - i_2) = E\{\nu(n+i_1)\nu^*(n+i_2)\}$ is the noise autocorrelation function (it is assumed that the noise is stationary, zero-mean). For white noise, $R_{\nu\nu}(i) = \sigma_\nu^2\delta(i)$, so we get:

$$E\{DSM_x(n,k)\} = DSM_f(n,k) +$$

$$\frac{\sigma_\nu^2}{2L_d + 1} \sum_{l=-L_d}^{L_d} \sum_{i=0}^{N-1} w^2(i)e^{-j\frac{2\pi}{N}2li} \qquad (13)$$

$$= DSM_f(n,k) + \frac{\sigma_\nu^2}{2L_d + 1} \sum_{l=-L_d}^{L_d} W_{w^2}(2l), \qquad (14)$$

where $W_{w^2}(l) = FT\{w^2(i)\}$ is the discrete Fourier transform of $w^2(i)$. Since, in the pseudo Wigner distribution, we have an equivalent window $w(i)w^*(-i)$ which, for a real and even window function, reduces to $w^2(i)$, the window $w^2(i)$ is used in the analysis instead of $w(i)$.

Relation (14) may be written as:

$$E\{DSM_x(n,k)\} =$$

$$DSM_f(n,k) + a_w\frac{N\sigma_\nu^2}{2L_d + 1}, \qquad (15)$$

where $a_w = \frac{1}{N}\sum_{l=-L_d}^{L_d} W_{w^2}(2l)$ is a constant depending on the selected window. For example, for the rectangular, Hanning and Hamming windows, this constant is 1, 0.5 and 0.54, respectively (see Table I).

A.2 Means in the spectrogram and Wigner distribution

For $L_d = 0$, i.e., assuming $P_d(l) = \delta(l)$, we have:

$$E\{DSM_x(n,k)\} = DSPEC_f(n,k) + Na_w\sigma_\nu^2, \qquad (16)$$

representing the mean value of the spectrogram, as obtained in [20] for the case of the rectangular window ($a_w = 1$).

Replacing $2L_d + 1 = N$ in equation (15), we get:

$$E\{DSM_x(n,k)\} = DPWD_f(n,k) + a_w\sigma_\nu^2. \qquad (17)$$

The above relation represents the mean of the Wigner distribution estimator. The same value is derived in [18]. One may conclude that relation (15) unifies the mean value expressions for the spectrogram and Wigner distribution.

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TABLE I

VARIANCE AND MEAN VALUE CONSTANTS

$w^2(i)$	Rectangular window	Hanning window	Hamming window
r_w	$(2L_d + 1)$	$(3L_d + 1)/4$	$(0.79L_d + 0.29)$
a_w	1	0.5	0.54

</div>

A.3 Variance in the S-method

Calculation of the DSM variance starts with the defining expression which, observing that $DSM_x(n,k)$ is a real function, may be written in the following form:

$$\sigma_{xx}^2 = \text{var}\{DSM_x(n,k)\} =$$

$$E\left\{DSM_x^2(n,k)\right\} - E^2\left\{DSM_x(n,k)\right\}. \quad (18)$$

After several routine manipulations, the variance in two components is achieved, as: $\sigma_{xx}^2 = \sigma_{f\nu}^2 + \sigma_{\nu\nu}^2$. Component $\sigma_{f\nu}^2$ depends both on signal $f(n)$ and on noise $\nu(n)$, while the other variance component $\sigma_{\nu\nu}^2$ depends exclusively on the additive noise $\nu(n)$.

The noise-only-dependent part of the variance is:

$$\sigma_{\nu\nu}^2 = \frac{1}{(2L_d + 1)^2} \times$$

$$\sum_{l_1=-L_d}^{L_d} \sum_{l_2=-L_d}^{L_d} \sum_{i_1=0}^{N-1} \sum_{i_2=0}^{N-1} \sum_{i_3=0}^{N-1} \sum_{i_4=0}^{N-1} w(i_1)w(i_2) \times$$

$$w(i_3)w(i_4)[E\{\nu(n+i_1)\nu^*(n+i_2) \times$$

$$\nu(n+i_3)\nu^*(n+i_4)\} - E\{\nu(n+i_1) \times$$

$$\nu^*(n+i_2)\}E\{\nu(n+i_3)\nu^*(n+i_4)\}] \times$$

$$\exp\left[-j\frac{2\pi}{N}k(i_1 - i_2 + i_3 - i_4)\right] \times$$

$$\exp\left[-j\frac{2\pi}{N}[l_1(i_1 + i_2) + l_2(i_3 + i_4)]\right]. \quad (19)$$

For Gaussian noise it holds [21]:

$$E\{\nu(n+i_1)\nu^*(n+i_2)\nu(n+i_3) \times$$

$$\nu^*(n+i_4)\} = R_{\nu\nu}(i_1 - i_2) \times$$

$$R_{\nu\nu}(i_3 - i_4) + R_{\nu\nu^*}(i_1 - i_3) \times$$

$$R_{\nu^*\nu}(i_2 - i_4) + R_{\nu\nu}(i_1 - i_4)R_{\nu\nu}(i_3 - i_2). \quad (20)$$

If we further assume that the noise is white (knowing that, for complex noise with independent real and imaginary parts, having

equal variances, $R_{\nu\nu^*}(i_1 - i_3) = R_{\nu^*\nu}(i_2 - i_4) = 0$), we get:

$$\sigma_{\nu\nu}^2 = \frac{\sigma_\nu^4}{(2L_d + 1)^2} \times$$

$$\sum_{l_1=-L_d}^{L_d} \sum_{l_2=-L_d}^{L_d} \sum_{i_1=0}^{N-1} \sum_{i_2=0}^{N-1} w^2(i_1)w^2(i_2) \times$$

$$\exp\left[-j\frac{2\pi}{N}[(l_1 + l_2)i_1 + (l_1 + l_2)i_2]\right]. \quad (21)$$

Finally, the distribution's variance may be written in a very simple and compact form as:

$$\sigma_{\nu\nu}^2 = \frac{r_w N^2 \sigma_\nu^4}{(2L_d + 1)^2}, \quad (22)$$

where r_w is a window-dependent constant:

$$r_w = \frac{1}{N^2} \sum_{l_1=-L_d}^{L_d} \sum_{l_2=-L_d}^{L_d} W_{w^2}^2(l_1 + l_2). \quad (23)$$

For example, for the rectangular, Hanning and Hamming windows, this constant is given in Table I. The normalized variance $\sigma_{\nu\nu}^2/(N^2\sigma_\nu^4)$ is shown in Figure 1.

On the basis of (22), we may easily write the variances in the pseudo Wigner distribution and spectrogram. For example, for the pseudo Wigner distribution and rectangular windows, substituting $2L_d + 1 = N$ into equation (22), we get:

$$\sigma_{\nu\nu}^2 = N\sigma_\nu^4,$$

while for the spectrogram, $L_d = 0$, we arrive at:

$$\sigma_{\nu\nu}^2 = N^2 \sigma_\nu^4.$$

The variance's component depending on both the signal and the noise is defined by:
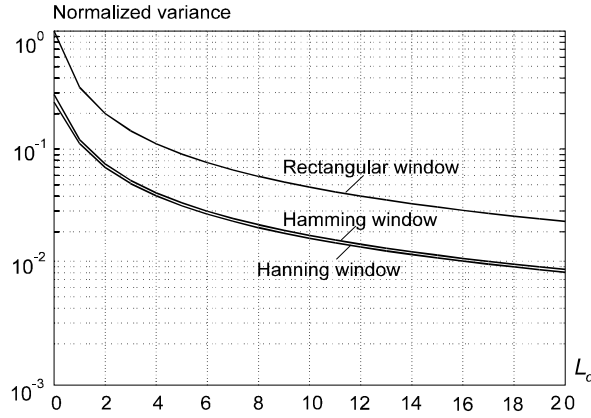
$$\sigma_{f\nu}^2 = \frac{1}{(2L_d + 1)^2} \times$$

Fig. 1. Normalized variance in the S-method

$$\sum_{l_1=-L_d}^{L_d} \sum_{l_2=-L_d}^{L_d} \sum_{i_1=0}^{N-1} \sum_{i_2=0}^{N-1} \sum_{i_3=0}^{N-1} \sum_{i_4=0}^{N-1} w(i_1) \times$$

$$w(i_2)w(i_3)w(i_4)\{f(n+i_1)f(n+i_3)\times$$

$$R_{\nu^*\nu}(i_2-i_4)+f(n+i_1)f^*(n+i_4)\times$$

$$R_{\nu\nu}(i_3-i_2)+f^*(n+i_2)f(n+i_3)\times$$

$$R_{\nu\nu}(i_1-i_4)+f^*(n+i_2)f^*(n+i_4)R_{\nu\nu^*}(i_1$$

$$-i_3)\}\exp\left[-j\frac{2\pi}{N}k(i_1-i_2+i_3-i_4)\right]\times$$

$$\exp\left[-j\frac{2\pi}{N}[l_1(i_1+i_2)+l_2(i_3+i_4)]\right]. \quad (24)$$

For complex white noise, with independent real and imaginary parts, having equal variances, the previous relation reduces to:

$$\sigma_{f\nu}^2 = \frac{\sigma_\nu^2}{(2L_d+1)^2}\times$$

$$\{\sum_{l_1=-L_d}^{L_d}\sum_{l_2=-L_d}^{L_d}\sum_{i_1=0}^{N-1}\sum_{i_3=0}^{N-1}\sum_{i_4=0}^{N-1}w^2(i_3)\times$$

$$w(i_1)f(n+i_1)w(i_4)f^*(n+i_4)\times$$

$$\exp[-j\frac{2\pi}{N}[i_1(k+l_1)-i_4(k-l_2)+$$

$$i_3(l_1+l_2)]]+$$

$$\sum_{l_1=-L_d}^{L_d}\sum_{l_2=-L_d}^{L_d}\sum_{i_1=0}^{N-1}\sum_{i_2=0}^{N-1}\sum_{i_3=0}^{N-1}w^2(i_1)\times$$

$$w(i_2)f^*(n+i_2)w(i_3)f(n+i_3)\times$$

$$\exp[-j\frac{2\pi}{N}[i_3(k+l_2)-i_2(k-l_1)+$$

$$i_1(l_1+l_2)]]\} \quad (25)$$

since the correlation functions $R_{\nu\nu^*}(i_1 - i_2) = E\{\nu(i_1)\nu(i_2)\} = 0$ and $R_{\nu^*\nu}(i_1 - i_2) = E\{\nu^*(i_1)\nu^*(i_2)\} = 0$ are equal to zero (note that for the complex zero-mean noise, with independent real $\nu_r(i)$ and imaginary $\nu_i(i)$ parts, we have $E\{\nu(i_1)\nu(i_2)\} = E\{\nu_r(i_1)\nu_r(i_2)\} - E\{\nu_i(i_1)\nu_i(i_2)\} = 0$).

The previous equation may be written in the form:

$$\sigma_{f\nu}^2 = \frac{2\sigma_\nu^2}{(2L_d+1)^2}\sum_{l_1=-L_d}^{L_d}\sum_{l_2=-L_d}^{L_d}W_{w^2}(l_1+l_2)\times$$

$$DSTFT_f(n,k+l_1)DSTFT_f^*(n,k-l_2). \quad (26)$$

For the rectangular window $w^2(i)$ ($W_{w^2}(l_1+l_2) = N\delta(l_1+l_2)$), this part of variance reduces to a very simple form:

$$\sigma_{f\nu}^2 = \frac{2N\sigma_\nu^2}{(2L_d+1)^2}\sum_{l=-L_d}^{L_d}DSPEC_f(n,k+l). \quad (27)$$

It may be concluded that this part of variance is heavily dependent on the spectrogram and window $P_d(l)$ width.

Very similar forms may be obtained for other window $w^2(i)$ forms. For example, for a Hanning window we easily get $W_{w^2}(l_1+l_2) = \frac{N}{2}[\delta(l_1+l_2) - (\delta(l_1+l_2+1) + \delta(l_1+l_2-1))/2]$ and the expression for variance $\sigma_{f\nu}^2$ similar to (27) follows.

B. Real noise

In the case of real noise $\nu(n)$, $R_{\nu\nu}(n) = R_{\nu^*\nu}(n) = R_{\nu\nu^*}(n) = R_{\nu^*\nu^*}(n)$, so we obtain the mean value as:

$$E\{DSM_x(n,k)\} = DSM_f(n,k) + a_w \frac{N\sigma_\nu^2}{2L_d+1}, \quad (28)$$

while the variance takes the form:

$$\sigma_{\nu\nu}^2 = \frac{\sigma_\nu^4}{(2L_d+1)^2} \times$$

$$\sum_{l_1=-L_d}^{L_d} \sum_{l_2=-L_d}^{L_d} \sum_{i_1=0}^{N-1} \sum_{i_2=0}^{N-1} w^2(i_1)w^2(i_2) \times$$

$$\left(1 + \exp\left[-j\frac{2\pi}{N}2k(i_1-i_2)\right]\right) \times$$

$$\exp\left[-j\frac{2\pi}{N}[(l_1+l_2)i_1 + (l_1+l_2)i_2]\right], \quad (29)$$

where the calculations, similar to those in Subsection III.1, are performed.

Appropriate transformations yield:

$$\sigma_{\nu\nu}^2 = \frac{r_w N^2 \sigma_\nu^4}{(2L_d+1)^2} + \frac{\sigma_\nu^4}{(2L_d+1)^2} \times$$

$$\sum_{l_1=-L_d}^{L_d} \sum_{l_2=-L_d}^{L_d} W_{w^2}(l_1+l_2+2k) \times$$

$$W_{w^2}(l_1+l_2-2k). \quad (30)$$

Having in mind the definitions of $W_{w^2}(l)$ and r_w, we arrive at a very simple expression:

$$\sigma_{\nu\nu}^2 = \frac{r_w N^2 \sigma_\nu^4}{(2L_d+1)^2}(1 + \delta(2k)), \quad (31)$$

which holds for the rectangular, Hanning and Hamming windows. The only difference between complex and real noise is in the existence of factor $\delta(2k)$, i.e., the variances in these two cases are the same except at frequencies which $k=0$, where the variance for the real noise case is twice greater.

The signal and noise-dependent part of the variance is:

$$\sigma_{f\nu}^2 = \frac{2\sigma_\nu^2}{(2L_d+1)^2} \sum_{l_1=-L_d}^{L_d} \sum_{l_2=-L_d}^{L_d} \{W_{w^2}(l_1+l_2) \times$$

$$DSTFT_f(n, k+l_1) \times$$

$$DSTFT_f^*(n, k-l_2) + W_{w^2}(l_1+l_2-2k) \times$$

$$\text{Re}\left[DSTFT_f(n, k+l_1)DSTFT_f(n, k+l_2)\right]\}. \quad (32)$$

One may conclude that the variances in the case of real noise are just slightly different than those in the case of complex noise with independent real and imaginary parts.

C. Variances in the spectrogram and Wigner distribution

The variances in the spectrogram and Wigner distribution (in exactly the same form as obtained in [20] and [18] with rectangular windows) follow from (31), noting that the delta pulse function may be written as:

$$\delta(k) = \frac{\sin\frac{2\pi}{N}kN}{N\sin\frac{2\pi}{N}k} = \left(\frac{\sin\frac{2\pi}{N}kN}{N\sin\frac{2\pi}{N}k}\right)^2. \quad (33)$$

Replacing $L_d = 0$ and $2L_d + 1 = N$, respectively, we get:

$$\sigma_{\nu\nu}^2 = N\sigma_\nu^4 \left(1 + \frac{\sin\frac{2\pi}{N}2kN}{N\sin\frac{2\pi}{N}2k}\right), \quad (34)$$

$$\sigma_{\nu\nu}^2 = \sigma_\nu^4 \left[N^2 + \left(\frac{\sin\frac{2\pi}{N}kN}{\sin\frac{2\pi}{N}k}\right)^2\right], \quad (35)$$

for the rectangular windows. Expressions (34) and (35) were separately obtained in [20] and [18]. If the noise is white, but not Gaussian, the results differ only slightly, [20], [18].

IV. EXAMPLES

Consider now (analytically and numerically) two simple examples:
1) sinusoidal noisy signal and,
2) linear frequency modulated noisy signal.

A. Example 1

Assume that the signal $f(n)$ inside a window, for a given instant n, may be treated as a sinusoid:

$$x(n) = f(n) + \nu(n) = Ae^{j\frac{2\pi}{N}k_0 n} + \nu(n). \quad (36)$$

In this case, the STFT at instant n is of the form:

$$DSTFT_f(n,k) = N\delta(k-k_0)f(n), \quad (37)$$

where a rectangular window $w(i)$ is used.

The S-method produces:

$$DSM_f(n,k) = \frac{1}{2L_d+1} \times$$

$$\sum_{l=-L_d}^{L_d} DSTFT_f(n,k+l)DSTFT_f^*(n,k-l) =$$

$$\frac{A^2N^2}{2L_d+1}\delta(k-k_0). \tag{38}$$

Two special cases, the spectrogram and the Wigner distribution, are:

$$DSPEC_f(n,k) = A^2N^2\delta(k-k_0),$$

$$DPWD_f(n,k) = A^2N\delta(k-k_0). \tag{39}$$

In order to investigate the influence of noise on the time-frequency representation, we define the *peak signal to noise ratio* ($S/N_{\max}$). It will be defined as the ratio of the squared absolute maximal value of the distribution and estimator's variance[1]. For the S-method, this ratio is:

$$S/N_{\max} = \frac{|\max\{DSM_f(n,k)\}|^2}{\sigma_{\nu\nu}^2 + \sigma_{f\nu}^2} \tag{40}$$

One may distinguish two parts of the time-frequency plane: one which will be denoted by $\Pi_{\nu\nu}$ where only $\sigma_{\nu\nu}^2$ exists, and the other $\Pi_{f\nu}$ where both components $\sigma_{\nu\nu}^2 + \sigma_{f\nu}^2$ exist.

In the $\Pi_{\nu\nu}$ region, the signal-to-noise ration is:

$$S/N_{\max} = \frac{\left(\frac{A^2N^2}{2L_d+1}\right)^2}{\frac{N^2\sigma_\nu^4}{2L_d+1}} = \frac{A^4N^2}{(2L_d+1)}. \tag{41}$$

[1]Another possible definition of the *signal-to-noise ratio* is the local ratio of distribution and its variance

$$S/N = \frac{|DSM_f(n,k)|^2}{\sigma_{\nu\nu}^2 + \sigma_{f\nu}^2}.$$

However, we preferred the definition (40) since it produces simpler results; it also compares the pick value of the distribution with the *noise* in the time-frequency plane. This is very reasonable in many practical applications, where a time-frequency distribution (its pick value(s)) is used to estimate the instantaneous frequency of a signal. In this case, we are not interested in the local ratio, especially at the points where the distribution is equal to zero. For that point, it is better to compare the variance, due to noise, with the maximum value of the distribution, since this ratio represents the measure of possible false peak detection (i.e., wrong frequency detection).

From the previous equation it may be concluded that the maximal S/N value is obtained for the spectrogram ($L_d = 0$), while the minimal value is obtained for the Wigner distribution ($2L_d + 1 = N$). This is an expected result, since it is common to consider the spectrogram as the smoothed Wigner distribution. However, the same results are not found in the case of the linear frequency modulated signal, which will be studied in the next example.

In the $\Pi_{f\nu}$ region, where both parts of the variance exist, the ratio is:

$$S/N_{\max} = \frac{(A/\sigma_\nu)^4}{\frac{2L_d+1}{N^2} + \frac{2A^2}{N\sigma_\nu^2}\sum_{l=-L_d}^{L_d}\delta(k-k_0+l)}. \tag{42}$$

In this region, the dominant factor is due to $\sigma_{f\nu}^2$. However, this factor exists only in the region defined by the window $P_d(l)$ width (Fig.2). For the spectrogram, region $\Pi_{f\nu}$ coincides with the domain where the spectrogram is different from zero.

In the Wigner distribution, the value ($S/N_{\max}$) contains variance $\sigma_{f\nu}^2$ for all frequencies, so we get:

$$S/N_{\max} = \frac{(A/\sigma_\nu)^4}{\frac{1}{N} + \frac{2}{N}(A/\sigma_\nu)^2}.$$

Assume, for example, that $A = \sigma_\nu$, then in the worst case (region $\Pi_{f\nu}$) for the spectrogram, we have $S/N_{\max} = N^2/(1+2N) \cong N/2$, while in the Wigner distribution $S/N_{\max} \cong N/3$. Obviously, the signal-to-noise ratio in these two distributions is of the same order. The difference is only $d = 10log(3/2) = 1.76[dB]$.

B. Example 2

In this example, a linear frequency modulated signal considered:

$$x(t) = f(t) + \nu(t) = Ae^{jat^2/2} + \nu(t). \tag{43}$$

The reason why we defined the signal in the analog, rather than in the discrete domain, is because we use some mathematical tools that are not well defined in the discrete form. Of course, we will transfer the results to the discrete domain before the noise analysis.
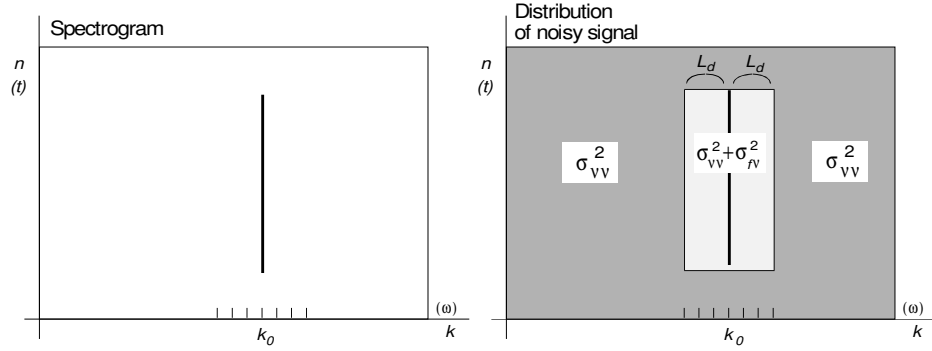
Fig. 2.   The distribution of variance in the case of a noisy sinusoidal signal

Assuming that constant a in (43) is large, one may use the stationary phase method[2] to obtain an approximate expression for the STFT:

$$STFT_f(t,\omega) \cong w\left(\frac{\omega - at}{a}\right) \times$$

$$\sqrt{\frac{2\pi j}{a}} e^{-j\omega^2/(2a)} f(t), \qquad (44)$$

or for the spectrogram:

$$SPEC_f(t,\omega) \cong \frac{2\pi A^2}{a} w^2\left(\frac{\omega - at}{a}\right). \quad (45)$$

[2]The stationary phase method [21] states that, for signal $x(t) = A(t)e^{j\varphi(t)}$, if $|A'(t)/A(t)| \ll |\phi'(t)|$, then:

$$X(\omega) = \int_{-\infty}^{\infty} A(t)e^{j\phi(t)} dt \cong$$

$$e^{j\phi(t_0)} A(t_0)\sqrt{2\pi j/|\phi''(t_0)|}$$

with $\phi'(t_0) = 0$, $\phi''(t_0) \neq 0$, and $\phi(t) = \varphi(t) - \omega t$. From the stationary phase method, it directly follows that, if we have a product of $x(t)$ and $w(t)$ (where $w(t)$ is slow-varying, i.e. $|[A(t)w(t)]'/[A(t)w(t)]| \ll |\phi'(t)|$ and if the instantaneous frequency may be treated as linear, i.e. $at_0 - \omega = 0$, then:

$$X_w(\omega) = \int_{-\infty}^{\infty} A(t)w(t)e^{j\phi(t)} dt = X(\omega)w(\omega/a)$$

where $w(\omega/a) = w(t)_{|t=\omega/a}$.

The discrete form[3] of spectrogram is:

$$DSPEC_f(n,k) \cong \frac{2\pi A^2 N^2}{aT^2} \times$$

$$w^2\left(\frac{2\pi k/T - anT/N}{a}\right), \qquad (46)$$

where T is the window $w(\tau)$ width and N is the number of samples.

The maximum possible value of the spectrogram (needed for the defined signal-to-noise ratio) is:

$$\max\{DSPEC_f(n,k)\} \cong \frac{2\pi A^2 N^2}{aT^2}. \quad (47)$$

The Wigner distribution of $f(t)$ is:

$$WD_f(t,\omega) = A^2 W(\omega - at),$$

$$\text{where } W(\omega) = FT\left\{w^2(\tau/2)\right\}, \qquad (48)$$

the maximum value of the discrete form Wigner distribution is:

$$\max\{DWD_f(n,k)\} = \frac{A^2 N}{T} C_w,$$

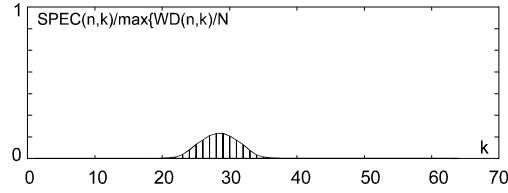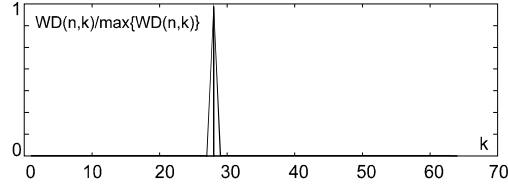$$\text{where } C_w = W(0) = \int_{-\infty}^{\infty} w^2(\tau/2) d\tau. \quad (49)$$

According to (8) and (49), we obtain:

$$\max\{DPWD_f(n,k)\} =$$

[3]The discrete form of the Fourier transform is:

$$\sum_{n=-\infty}^{\infty} x(n\triangle t)e^{-j\omega n\triangle t} \cong \frac{1}{\triangle t}\int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt,$$

where $\triangle t$ is sampling interval, $\triangle t = T/N$.

Fig. 3. Normalized spectrogram at a given instant t

Fig. 4. Normalized Wigner distribution at a given instant t

$$\max\{\frac{1}{N}\sum_{l=-N/2}^{N/2}\alpha(l)DSTFT_f(n,k+l)\times$$

$$DSTFT_f^*(n,k-l)\} = \frac{A^2N}{T}C_w. \qquad (50)$$

If the width of the STFT along the frequency axis (i.e. the width of $w((2\pi k/T - anT/N)/a)$) is equal to $2L_{dm}+1$, then the maximum value of the S-method, according to (50), is:

$$\max\{DSM_f(n,k)\} =$$

$$\max\{\frac{1}{2L_{dm}+1}\sum_{l=-L_{dm}}^{L_{dm}} DSTFT_f(n,k+l)\times$$

$$DSTFT_f^*(n,k-l)\} = \frac{N}{2L_{dm}+1}\frac{A^2N}{T}C_w. \tag{51}$$

Now, we may find the $S/N_{\max}$ ratio. Consider first the part of the time-frequency plane where only $\sigma_{\nu\nu}^2$ exists. For the spectrogram, in $\Pi_{\nu\nu}$ region, we get:

$$S/N_{\max} = \frac{\max\{DSPEC_f(n,k)\}^2}{\sigma_{\nu\nu}^2} =$$

$$\frac{\left(\frac{2\pi A^2N^2}{aT^2}\right)^2}{N^2\sigma_\nu^4} = \frac{4\pi^2N^2A^4}{(aT^2)^2\sigma_\nu^4}. \qquad (52)$$

For the S-method with $L_d \geq L_{dm}$ (including the Wigner distribution with $2L_d+1 = N$), it follows:

$$S/N_{\max} = \frac{\max\{DSM_f(n,k)\}^2}{\sigma_{\nu\nu}^2} =$$

$$\frac{\left(\frac{N}{2L_d+1}\frac{A^2N}{T}C_w\right)^2}{\frac{N^2\sigma_\nu^4}{2L_d+1}} = \frac{N^2A^4C_w^2}{T^2(2L_d+1)\sigma_\nu^4},$$

$$\text{for } L_d \geq L_{dm.} \qquad (53)$$

The ratio of the signal-to-noise ratios in the spectrogram and in the S-method (relations (53) and (52)) is:

$$R = \frac{S/N_{\max}|_{S-method}}{S/N_{\max}|_{spectrogram}} =$$

$$\left(\frac{aT^2}{\pi}\right)^2\frac{C_w^2}{4T^2}\frac{1}{2L_d+1} = r_s r_w r_p. \qquad (54)$$

We see that there are three parts of the coefficient R: r_s, depending on the signal form (its parameter a); r_w, depending on the window $w(\tau)$ and r_p depending on the window $P_d(l)$. Coefficient R will be used for the comparison of the S-method and spectrogram with respect to its maximal squared absolute values and the variance of the distribution (due to noise only) in the time-frequency plane. If this ratio is greater than 1, it means that the S-method has greater *peack signal-to-noise ratio*, i.e., that

the estimation of the instantaneous frequency (based on the S-method) is more reliable. We may easily conclude that R decreases as L_d increases. Maximum value of R is reached for the minimum value $L_d = L_{dm}$, for which relation (54) holds. In order to achieve that value of R, note that the maximum frequency, sampling period along the frequency, and L_{dm} are given by:

$$\omega_m = \pi N/T;$$

$$\triangle\omega = 2\pi/T;$$

$$2L_{dm} + 1 = aT/\triangle\omega = aT^2/2\pi.$$

Note that L_{dm} is obtained according to (45), (46). Thus, a very simple expression for the maximum R is obtained:

$$R = \frac{a}{2\pi}C_w^2. \tag{55}$$

In the $\Pi_{f\nu}$ part of time-frequency plane, where both components of the variance exist, the expressions for the signal-to-noise ratios are slightly more complex. For the spectrogram, we have:

$$S/N_{\max} = \frac{((2\pi A^2 N^2)/(aT^2))^2}{\frac{4\pi A^2 \sigma_\nu^2 N^3}{aT^2} + N^2\sigma_\nu^4}.$$

Note, again, that this region coincides with that in which the spectrogram is different from zero. While, for the S-method, signal-to-noise rations is:

$$S/N_{\max} = \frac{\left(\frac{N}{2L_d+1}\frac{A^2N}{T}C_w\right)^2}{\sigma_{f\nu}^2 + \frac{N^2\sigma_\nu^4}{2L_d+1}}$$

$$\geq \frac{\left(\frac{N}{2L_d+1}\frac{A^2N}{T}C_w\right)^2}{\frac{4\pi A^2\sigma_\nu^2 N^3}{aT^2(2L_d+1)} + \frac{N^2\sigma_\nu^4}{2L_d+1}},$$

$$\text{for } L_d \geq L_{dm}.$$

C. Numerical example 1

In this numerical example we considered:

$$x(t) = Ae^{jat^2/2} + \nu(t),$$

with $a = 1400$, the Hanning window of the width $T = 0.25$, number of samples $N = 128$, amplitude $A = 1$ and noise variance $\sigma_\nu^2 = 1$.

Results with spectrogram, S-method ($L_d = 2$), S-method ($L_d = 6$) and the Wigner distribution are shown in Figures 5a,b,c,d and 6a,b,c,d, respectively (with, Fig.6, and without noise, Fig.5). The coefficient R (equation (55)), in this case, is $R = 7.85$ or the ratio of amplitudes $\sqrt{R} = \sqrt{7.85} = 2.82$. This coefficient analytically proves (as is visually obvious from Figure 6) that the S-method may significantly improve the representation, in a noisy environment, with respect to the spectrogram and the Wigner distribution.

Note that the noise in spectrogram is especially pronounced just inside the region where the spectrogram is different from zero ($\Pi_{f\nu}$ region) (Fig.6a). In Figure 6c, the S-method with $L_d = 6$ (which is sufficient to obtain the same signal representation as in the Wigner distribution) is presented. The region $\Pi_{f\nu}$ is spread around the Wigner distribution in the region defined by $L_d = 6$. In the remaining part of the time-frequency plane, the noise is less pronounced. A variant between the cases in Figure 6a and Figure 6c is presented in Figure 6b, where the distribution is almost concentrated at the instantaneous frequency, with a very narrow $\Pi_{f\nu}$ region defined by $L_d = 2$. The case of the Wigner distribution where the region $\Pi_{f\nu}$ is the entire time-frequency plane is presented in Figure 6d. The advantage of the S-method with a small L_d, in the time frequency analysis of the noisy signals, is evident from Figure 6, as well as from the analytical treatment perfomed in the previous sections.

D. Numerical example 2

Time-frequency representation of a multi-component signal:

$$x(t) = f(t) + \nu(t) = e^{j1400t} + e^{j680(t-0.1)^2} + \nu(t),$$

is presented in Figure 7. The variance of white Gaussian noise is $\sigma_\nu^2 = 1$. The same number of samples and window $w(i)$ are used as in Figures 5 and 6. From Figure 7, we may conclude that the S-method, with $L_d = 3$, achieves almost the same concentration as the Wigner distribution, but the noise influence is significantly decreased with respect to the Wigner distribution. Also, in contrast to the Wigner distribution, the S-method is cross-term free
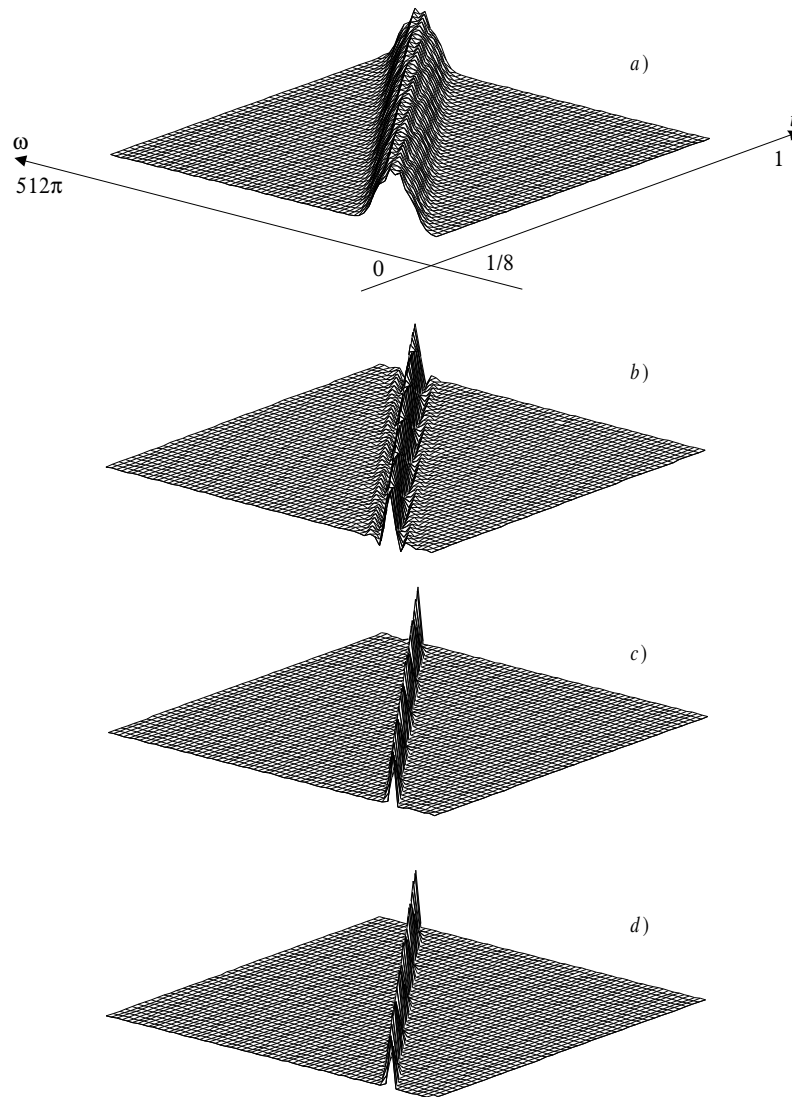
Fig. 5. Time-frequency representation of linear frequency modulated signal without noise: a) Spectrogram, b) S-method with $L_d = 2$, c) S-method with $L_d = 6$, and d) The Wigner distribution.

(Figure 7). This figure further demonstrates the above-described properties concerning sinusoidal and linear frequency modulated signals.

An interesting system with a signal-dependent window $P_d(l)$ width which, in the case of multicomponent signals, follows the components' widths, and stops all summations in (9) outside the auto-therm, is presented in [23], while an example with a real seismic signal is given in [24].

V. CONCLUSION

Noise analysis in the spectrogram and Wigner distribution is perfomed using the S-method. It is shown that the results for these two very important distributions readily follow as special cases from the S-method analysis. Also, in the case when the frequency is not constant, the S-method enables an improvement of the time-frequency presentation as compared to its two marginal cases, the
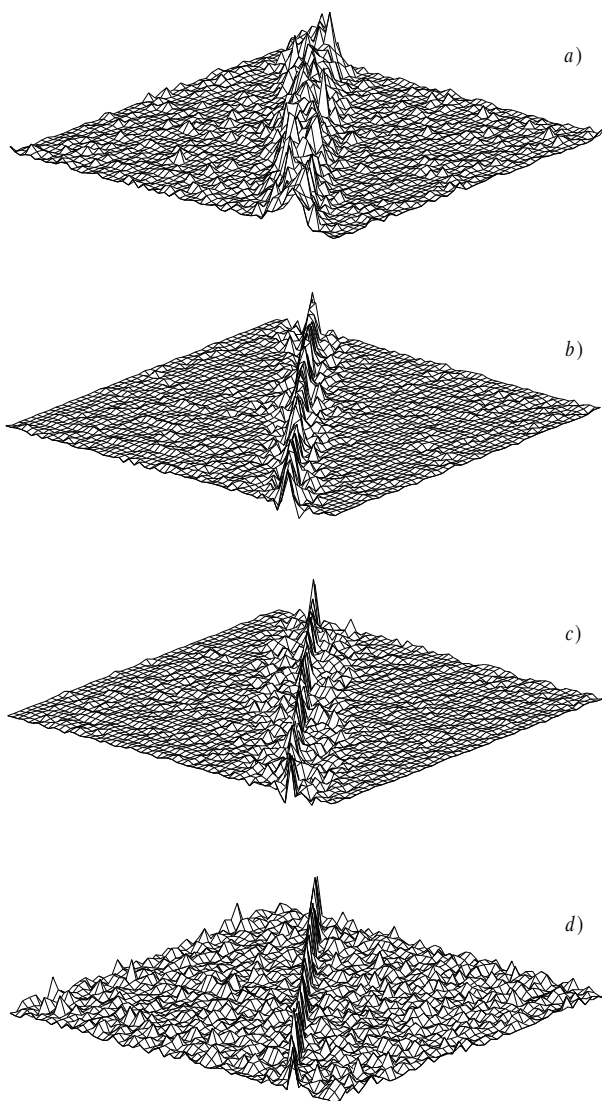
Fig. 6. Time-frequency representation of linear frequency modulated signal with noise ($SNR = 10\log(A^2/\sigma_\nu^2) = 0$ [dB]): a) Spectrogram, b) S-method with $L_d = 2$, c) S-method with $L_d = 6$, and d) The Wigner distribution.

spectrogram and Wigner distribution.

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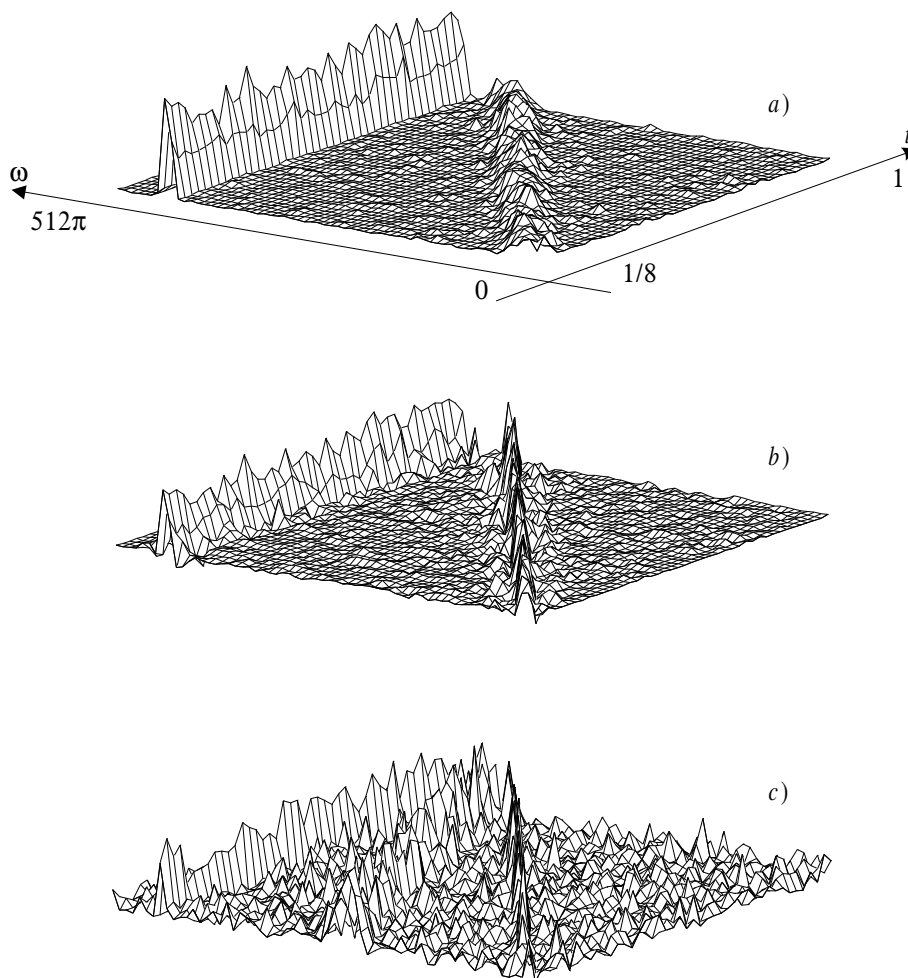
Fig. 7. Time-frequency representation of a multicomponent noisy signal: a) Spectrogram, b) S-method with $L_d = 3$, c) Wigner distribution.

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