Influence of interference on range estimation in noise radar systems

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Abstract— **The problem of false range estimation in noise radars operating in interference environment is investigated. A closed-form expression for the probability of false range estimation when the received signal is corrupted by interference with arbitrary phase is derived and it contains elementary mathematical functions only. Simulations support the derived probability. The expression can be easily adapted for the case when the interference is not present.**

I. Introduction

Noise radar has a number of advantages over conventional radars. Some of advantages are unambiguous range estimation, high range resolution, inherent anti-jamming capabilities, low probability of detection and interception. Wide bandwidth provides a high range resolution, while an extended pulse length reduces the peak power which further supports low probability of detection. A non-periodic waveform suppresses the range ambiguity while reducing both the probability of intercept and influence of interferences. In the recent past, a significant research has been done on the development and implementation of random noise radar [1–5]. A potential use of noise radar for the ultrawide-band SAR/ISAR imaging has been investigated, as well as for Doppler and polarimetric measurements, collision warning, detection of buried objects and targets obscured by foliage.

In noise radars, the received signal is correlated with delayed versions of the transmitted noise. From the position of the correlation peak, we can estimate the target's range. If, however, the received signal is corrupted by a strong interference, the variance of the correlation will be increased so that false peaks, that exceed the true one, can occur. In this paper, we derive a closed-form expression for the probability of false range estimation in interference environment. We will assume that the interference is a phase-modulated signal with arbitrary phase function.

The paper is organized as follows. Section II covers the theoretical background regarding the noise radar. The probability of false range estimation is derived in Section III, and numerically verified in Section IV. Conclusions are drawn in Section V.

II. Noise radar

In noise radar systems, random noise signal is transmitted, reflected from a target, and received with a delay $T_d = \frac{2r}{c}$, where r and c denote the target's range and the speed of light, respectively. The received signal is correlated with a replica of the transmitted noise delayed by T_r. A strong correlation peak is received when $T_d = T_r$ and its position provides an estimate of the target's range.

Let us consider a radar transmitting a complex random noise $x(t)$ given by

$$x(t) = x_R(t) + jx_I(t), \tag{1}$$

where both real and imaginary part, $x_R(t)$ and $x_I(t)$, respectively, are zero-mean bandlimited Gaussian processes with bandwidth B and variance $\frac{\sigma_x^2}{2}$. We will assume that a single point scatterer is located at the range r along the radar's line-of-sight. The received signal $y(t)$ can be modeled as

$$y(t) = A_\sigma x(t - T_d) + \xi(t), \tag{2}$$

where A_σ is the target's reflectivity, $T_d = \frac{2r}{c}$ is the round-trip delay and $\xi(t)$ is the additive ambient noise which will be modeled as complex white Gaussian noise as follows:

$$\xi(t) = \xi_R(t) + j\xi_I(t), \tag{3}$$

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where $\xi_R(t)$ and $\xi_I(t)$ have zero mean and variance $\frac{\sigma_\xi^2}{2}$. Signals $x(t)$ and $\xi(t)$ are uncorrelated with each other. Without loss of generality, we will take $A_\sigma = 1$. The correlation between the received and delayed transmitted signal is

$$C(T_r) = \int_0^{T_{int}} y(t)x^*(t - T_r)dt, \tag{4}$$

where T_{int} is the integration time or the pulse duration. It can be shown that [4]

$$E[C(T_r)] = T_{int}R_{xx}(T_r - T_d),$$

where $E[\cdot]$ denotes the statistical expectation and $R_{xx}(\tau)$ is the autocorrelation function of $x(t)$. Since $|R_{xx}(\tau)| \le R_{xx}(0)$ [6], the delay T_d can be estimated as the position of the maximum of $|C(T_r)|$.

In discrete time, the correlation $C(T_r)$ can be analyzed by describing the correlation integral as [4]

$$\begin{aligned} C(n_r) &= \sum_{n=1}^{N} y(n)\, x^*(n - n_r) \\ &= \sum_{n=1}^{N} [x(n - n_d) + \xi(n)]\, x^*(n - n_r), \end{aligned} \tag{5}$$

where integers n_d and n_r correspond to T_d and T_r, respectively, and N represents the number of samples that correspond to T_{int}. Again, a strong correlation peak occurs when $n_r = n_d$.

III. Probability of false range estimation

We will assume that the received signal $y(n)$ is corrupted by an interference $I(n)$,

$$y(n) = x(n - n_d) + I(n) + \xi(n). \tag{6}$$

All the components of $y(n)$ are uncorrelated with each other. In addition, we will adopt the following form of the interference:

$$I(n) = A_I e^{j\phi(n)}, \tag{7}$$

where A_I is constant amplitude and $\phi(n)$ is the phase of the interference. Signal-to-noise ratio (SNR) and signal-to-interference ratio (SIR) are defined as

$$\mathrm{SNR} = \frac{\sigma_x^2}{\sigma_\xi^2} \tag{8}$$

$$\mathrm{SIR} = \frac{\sigma_x^2}{A_I^2}. \tag{9}$$

When $n_r = n_d$ we have

$$C(n_d) = \sum_{n=1}^{N} [x(n - n_d) + I(n) + \xi(n)] \times x^*(n - n_d),$$

and the real and imaginary part of $C(n_d)$ satisfy

$$\begin{aligned} \mathrm{Re}[C(n_d)] &= \sum_{n=1}^{N} \left(x_R^2(n - n_d) + x_I^2(n - n_d)\right) \\ &+ \sum_{n=1}^{N} x_R(n - n_d)[A_I\cos(\phi(n)) + \xi_R(n)] \\ &+ x_I(n - n_d)[A_I\sin(\phi(n)) + \xi_I(n)] \\ \mathrm{Im}[C(n_d)] &= \sum_{n=1}^{N} x_R(n - n_d) \\ &\times [A_I\sin(\phi(n)) + \xi_I(n)] \\ &- x_I(n - n_d)[A_I\cos(\phi(n)) + \xi_R(n)]. \end{aligned}$$

According to the central limit theorem, both $\mathrm{Re}[C(n_d)]$ and $\mathrm{Im}[C(n_d)]$ are normally distributed, and it can be straightforwardly shown that

$$\begin{aligned} E\{\mathrm{Re}[C(n_d)]\} &= N\sigma_x^2 \\ E\{\mathrm{Im}[C(n_d)]\} &= 0 \\ \mathrm{Var}\{\mathrm{Re}[C(n_d)]\} &= N\sigma_x^4 + \frac{N}{2}\left(A_I^2 + \sigma_\xi^2\right)\sigma_x^2 \\ &= \sigma_x^4\frac{N}{2}\left(2 + \frac{1}{\mathrm{SIR}} + \frac{1}{\mathrm{SNR}}\right) \\ &\approx \sigma_x^4\frac{N}{2}\left(\frac{1}{\mathrm{SIR}} + \frac{1}{\mathrm{SNR}}\right) \\ \mathrm{Var}\{\mathrm{Im}[C(n_d)]\} &= \frac{N}{2}\left(A_I^2 + \sigma_\xi^2\right)\sigma_x^2 \\ &= \sigma_x^4\frac{N}{2}\left(\frac{1}{\mathrm{SIR}} + \frac{1}{\mathrm{SNR}}\right), \end{aligned}$$

where Var$\{\cdot\}$ represents the variance operator. In this paper, we will assume that

$$\frac{1}{\text{SNR}}+\frac{1}{\text{SIR}}\gg 2, \tag{10}$$

so the approximation in the expression for $\text{Var}\{\text{Re}[C(n_d)]\}$ holds. Recall that extended pulse length, i.e., bigger N, reduces the peak power, which, in turn, allows for smaller SNR and SIR values.

On the other hand, when $n_r \neq n_d$, samples $x_R(n-n_r)$ and $x_R(n-n_d)$ are statistically independent [4,5], and we have

$$\begin{aligned}
\text{Re}[C(n_r)] &= \sum_{n=1}^{N} x_R(n-n_r) \\
&\times [x_R(n-n_d)+A_I\cos(\phi(n))+\xi_R(n)] \\
&+ x_I(n-n_r) \\
&\times [x_I(n-n_d)+A_I\sin(\phi(n))+\xi_I(n)] \\
\text{Im}[C(n_r)] &= \sum_{n=1}^{N} x_R(n-n_r) \\
&\times [x_I(n-n_d)+A_I\sin(\phi(n))+\xi_I(n)] \\
&- x_I(n-n_r) \\
&\times [x_R(n-n_d)+A_I\cos(\phi(n))+\xi_R(n)].
\end{aligned}$$

Both Re$[C(n_r)]$ and Im$[C(n_r)]$ are normally distributed with

$$\begin{aligned}
E\{\text{Re}[C(n_r)]\} &= 0 \\
E\{\text{Im}[C(n_r)]\} &= 0 \\
\text{Var}\{\text{Re}[C(n_r)]\} &= \frac{N}{2}\left(\sigma_x^4+\sigma_x^2 A_I^2+\sigma_x^2\sigma_\xi^2\right) \\
&= \sigma_x^4\frac{N}{2}\left(1+\frac{1}{\text{SIR}}+\frac{1}{\text{SNR}}\right) \\
&\approx \sigma_x^4\frac{N}{2}\left(\frac{1}{\text{SIR}}+\frac{1}{\text{SNR}}\right) \\
\text{Var}\{\text{Im}[C(n_r)]\} &= \text{Var}\{\text{Re}[C(n_r)]\}.
\end{aligned}$$

Note that the variances of real and imaginary parts of $C(n_d)$ and $C(n_r)$ coincide.

The probability of false range estimation, denoted as P_{FRE}, equals the probability that $|C(n_d)|$ does not reach maximum for $n_r = n_d$ or it does not exceed a predefined threshold T. For the sake of simplicity, we will denote the variable $|C(n_r)|$ for $n_r \neq n_d$ as C_r and $|C(n_d)|$ as C_d. Since Re$[C(n_r)]$ and Im$[C(n_r)]$ are Gaussian random variables with zero mean and common variance, C_r is a Rayleigh variable, characterized by the probability density function (p.d.f.) [6]

$$f_{C_r}(x) = \frac{x}{\sigma_{C_r}^2} e^{-\frac{x^2}{2\sigma_{C_r}^2}} U(x), \tag{11}$$

with the scale parameter

$$\begin{aligned}
\sigma_{C_r}^2 &= \text{Var}\{\text{Re}[C(n_r)]\} \\
&= \sigma_x^4\frac{N}{2}\left(\frac{1}{\text{SIR}}+\frac{1}{\text{SNR}}\right)
\end{aligned} \tag{12}$$

and $U(x)$ is the unit step function

$$U(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0. \end{cases}$$

On the other hand, since Re$[C(n_d)]$ and Im$[C(n_d)]$ are Gaussian random variables with common variance and different means, C_d will be a Rician variable with the p.d.f. [6]

$$f_{C_d}(x) = \frac{x}{\sigma_{C_d}^2} e^{-\frac{x^2+\mu_{C_d}^2}{2\sigma_{C_d}^2}} I_0\left(\frac{\mu_{C_d}}{\sigma_{C_d}^2}x\right) U(x), \tag{13}$$

where the Rician distribution parameters μ_{C_d} and $\sigma_{C_d}^2$ equal

$$\mu_{C_d} = E\{\text{Re}[C(n_d)]\} = N\sigma_x^2 \tag{14}$$

$$\begin{aligned}
\sigma_{C_d}^2 &= \text{Var}\{\text{Re}[C(n_d)]\} \\
&= \sigma_x^4\frac{N}{2}\left(\frac{1}{\text{SIR}}+\frac{1}{\text{SNR}}\right),
\end{aligned} \tag{15}$$

and $I_0(x)$ is the modified Bessel function of the first kind and zeroth order

$$I_0(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos(\theta)}d\theta.$$

Instead of calculating P_{FRE}, we will calculate the probability of correct range estimation, denoted as P_{CRE}, which equals the probability that the Rician variable C_d exceeds $N-1$ independent Rayleigh variables C_r, and, at the same time, exceeds a threshold T.

By definition, the distribution function of the random variable C_r, $F_{C_r}(x)$, equals the probability that C_r does not exceed x. Having this in mind, probability that $N-1$ independent variables C_r do not exceed x equals $F_{C_r}^{N-1}(x)$, which, in turn, implies that

$$P_{CRE} = \int_T^{+\infty} F_{C_r}^{N-1}(x) f_{C_d}(x)\, dx. \qquad (16)$$

Substituting $F_{C_r}(x)$ for the Rayleigh distribution,

$$F_{C_r}(x) = \left(1 - e^{-\frac{x^2}{2\sigma_{C_r}^2}}\right) U(x),$$

and (13) into (16), we obtain

$$\begin{aligned} P_{CRE} &= \int_T^{+\infty} \left(1 - e^{-\frac{x^2}{2\sigma^2}}\right)^{N-1} \\ &\times \frac{x}{\sigma^2} e^{-\frac{x^2+\mu^2}{2\sigma^2}} I_0\left(\frac{\mu}{\sigma^2}x\right) U^N(x)\, dx \\ &= \int_T^{+\infty} \left(1 - e^{-\frac{x^2}{2\sigma^2}}\right)^{N-1} \\ &\times \frac{x}{\sigma^2} e^{-\frac{x^2+\mu^2}{2\sigma^2}} I_0\left(\frac{\mu}{\sigma^2}x\right) dx, \qquad (17) \end{aligned}$$

where σ and μ satisfy

$$\sigma = \sigma_{C_r} = \sigma_{C_d} \qquad (18)$$

$$\mu = \mu_{C_d}. \qquad (19)$$

Taking into account the binomial theorem, i.e.

$$\left(1 - e^{-\frac{x^2}{2\sigma^2}}\right)^{N-1} = \sum_{m=0}^{N-1} \binom{N-1}{m} (-1)^m e^{-\frac{mx^2}{2\sigma^2}}$$

and the following series representation of $I_0(x)$ [7, p.919]:

$$I_0(x) = \sum_{k=0}^{+\infty} \frac{\left(\frac{x}{2}\right)^{2k}}{(k!)^2},$$

we can rewrite P_{CRE} as

$$\begin{aligned} P_{CRE} &= \frac{e^{-\frac{\mu^2}{2\sigma^2}}}{\sigma^2} \sum_{m=0}^{N-1} \binom{N-1}{m} (-1)^m \\ &\times \sum_{k=0}^{+\infty} \frac{\left(\frac{\mu}{2\sigma^2}\right)^{2k}}{(k!)^2} \int_T^{+\infty} x^{2k+1} e^{-\frac{(m+1)x^2}{2\sigma^2}} dx. \qquad (20) \end{aligned}$$

The integral in (20) can be calculated using [7, p.346]

$$\int_T^{+\infty} x^m e^{-\beta x^n} dx = \frac{\Gamma\left(\frac{m+1}{n}, \beta T^n\right)}{n\beta^{\frac{m+1}{n}}},$$

where $T > 0, \beta > 0, m > 0, n > 0$, and $\Gamma(\alpha, x)$ is the incomplete Gamma function. Straightforwardly

$$\begin{aligned} &\int_T^{+\infty} x^{2k+1} e^{-\frac{(m+1)x^2}{2\sigma^2}} dx = \\ &= 2^k \frac{\sigma^{2k+2}}{(m+1)^{k+1}} \Gamma\left(k+1, \frac{m+1}{2\sigma^2}T^2\right) \\ &= 2^k \frac{\sigma^{2k+2}}{(m+1)^{k+1}} k! e^{-\frac{m+1}{2\sigma^2}T^2} \sum_{p=0}^{k} \frac{\left(\frac{m+1}{2\sigma^2}T^2\right)^p}{p!}. \end{aligned} \qquad (21)$$

In (21), we have used the property of the incomplete Gamma function [7, p.899]

$$\Gamma(k+1, x) = k! e^{-x} \sum_{p=0}^{k} \frac{x^p}{p!}, \qquad k = 0, 1, 2, \cdots$$

The probability of correct range estimation finally becomes

$$\begin{aligned} P_{CRE} &= e^{-\frac{\mu^2}{2\sigma^2}} \sum_{m=0}^{N-1} (-1)^m \binom{N-1}{m} e^{-\frac{m+1}{2\sigma^2}T^2} \\ &\times \sum_{k=0}^{+\infty} \frac{\left(\frac{\mu^2}{2\sigma^2}\right)^k}{k!\,(m+1)^{k+1}} \sum_{p=0}^{k} \frac{\left(\frac{m+1}{2\sigma^2}T^2\right)^p}{p!} \end{aligned} \qquad (22)$$

and the probability of false range estimation

$$P_{FRE} = 1 - P_{CRE}. \qquad (23)$$

The expression for P_{CRE} contains elementary functions only, and is, therefore, easy to be calculated.

In specific, the probability of false range estimation when the interference is not present in the received signal is obtained by setting $\frac{1}{\mathrm{SIR}} = 0$ in the expression for σ (see (12) and (18)).

Note that, when $T = 0$, P_{CRE} represents the probability that a Rician variable exceeds N Rayleigh variables, where all variables are characterized by the same parameter σ. This probability can be derived from (17) by replacing the lower integration limit with $-\infty$.

A. Threshold selection

We will calculate the threshold T according to the Neyman-Pearson criterion, i.e., for a given probability of false alarm, P_{FA}. When no radar return is present in the received signal, the variable C_d is a Rayleigh variable with the scale parameter σ (18). The probability of false alarm therefore equals

$$P_{FA} = \int_{T}^{+\infty} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx, \tag{24}$$

which yields the following threshold

$$T = \sqrt{2}\sigma\sqrt{-\ln P_{FA}}. \tag{25}$$

IV. Simulations

Let us consider a noise radar operating with the bandwidth $B = 204.8$ MHz and the pulse duration of $T_{int} = 10\mu s$. We will assume that the sampling rate, at baseband, equals the Nyquist rate $T_s = 1/B$; therefore one pulse contains $N = 2048$ samples.

First, we will calculate the probability of false range estimation versus SIR for several SNRs. In specific, the SIR is varied from -30dB to -10dB in increments of 2dB. Four different SNRs are considered, -20dB, -15dB, -10dB and -5dB. The results are shown in Fig.1, where solid lines and diamonds represent analytical and numerical results, respectively. Clearly, analytical results are completely confirmed by the numerical ones. The interference is a third-order polynomial phase signal, whose phase is given by

$$\phi(n) = 2\pi\left(-\frac{N}{5}(n\Delta) - \frac{N}{4}(n\Delta)^2 + \frac{N}{3}(n\Delta)^3\right),$$

where $\Delta = \frac{1}{N}$ and $n = 0, 1, \cdots, N-1$.

The probability of false range estimation versus SNR curve for the interference-free case is depicted in Fig.2, where SNR is varied from -30dB to -12dB in increments of 2dB. In addition, curves for $N = 1024$ ($T_{int} = 5\mu s$) and $N = 512$ ($T_{int} = 2.5\mu s$) are depicted in Fig.2.

In all the examples, the threshold T is calculated so that $P_{FA} = 10^{-5}$ holds.

In this section, the analytical results have been obtained using the Mathematica software, whereas the numerical ones have been obtained using Matlab.

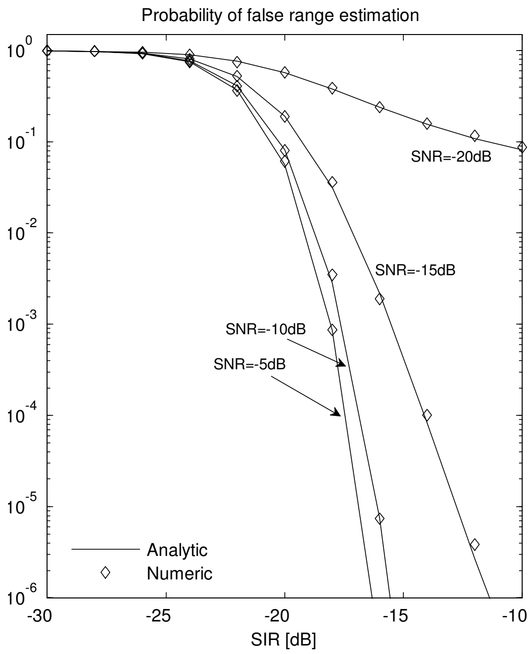

Fig. 1. Probability of false range estimation versus SIR. Four values of SNR are considered.

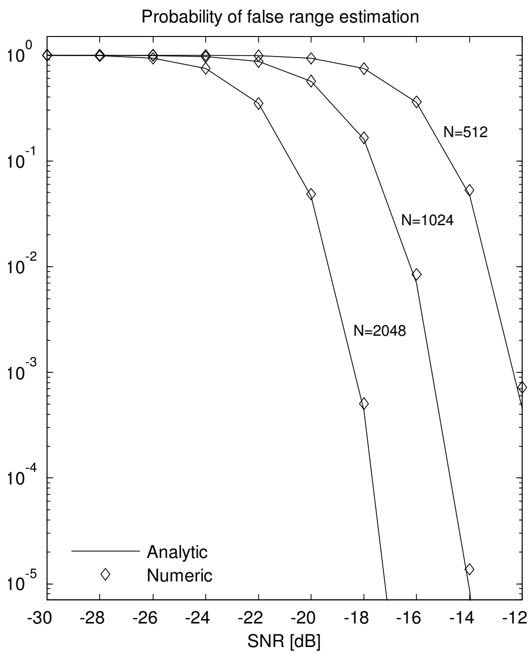

Fig. 2. Probability of false range estimation versus SNR. Results for $N = 1024$ and $N = 512$ are also given.

V. Conclusion

In this paper, we addressed the problem of false range estimation in noise radars. We considered the received signal corrupted by interference that has the form of a phase-modulated signal. A closed-form expression for the probability of false range estimation is derived and numerically supported. A distinguishable feature of the expression is that it does not contain special functions which makes it easy to be calculated. The expression can be also used in the special case when the interference is not present. The probability that a Rician variable exceeds N Rayleigh variables can be easily obtained from the derived probability.

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