point ω_i and θ_k, i.e.

$$\psi(\omega_i, \theta_k) = \sum_{m=-N/2}^{N/2} \sum_{\tau=-N/2}^{N/2-1} \phi(m, \tau) e^{-j(\omega_i \tau + 2\theta_k m)} \qquad (24)$$

where $N = 2L_{\max}$, and

$$\theta_k = \frac{2\pi k}{N}, \omega_i = \frac{2\pi i}{N}, \quad \text{where } i, k \in \{-N/2, \cdots, N/2-1\}.$$

Then

$$\psi(\omega_i, \theta_k) = 0 \quad \text{for} \quad |\omega_i| > |\theta_k|. \qquad (25)$$

Equation (24) can be computed using the DFT applied to Hermitian autocorrelation functions as proposed in [10]. If the frequency support constraint is included in the problem, minimization of the cost function (9) can no longer be performed over each lag separately. In this case, the solution can be obtained via matrix formulation of the problem [11], [12], in which the rows of the kernel are stacked to form a vector. With this formulation, the problem becomes

$$\text{Minimize } \boldsymbol{w}^H \boldsymbol{w} \quad \text{subject to } \boldsymbol{C}^H \boldsymbol{w} = \boldsymbol{f}. \qquad (26)$$

In the above notation, "H" denotes Hermitian, the variance (9) equals $\boldsymbol{w}^H \boldsymbol{w}$, and the t-f linear constraints are represented by the constraint system $\boldsymbol{C}^H \boldsymbol{w} = \boldsymbol{f}$, where $\boldsymbol{C}$ and $\boldsymbol{f}$ are, respectively, referred to as the constraint matrix and the gain vector. The linear constraints are not all independent, thus $\boldsymbol{C}$ is not of full rank. The solution is, therefore, given by [13]

$$\boldsymbol{w} = (C^{\#})^H \boldsymbol{f} \qquad (27)$$

where $C^{\#}$ is the pseudo-inverse of $\boldsymbol{C}$.

The optimum kernel satisfying (26) is referred to as the minimum variance kernel. This kernel is shown in Fig. 2 in the time-lag domain, the ambiguity domain, and the frequency-frequency domain, for $L_{\max} = 28$.

Next, we presents a simulation example that illustrates the kernels' statistical performance for the case of a sinusoid in noise. The time-frequency distributions of several kernels have been simulated for a complex sinusoid of fixed frequency in circular complex white noise. The sinusoid is of 0.5 dB and 100 Hz with the sampling frequency 1 kHz. For each kernel, 20 realizations of the time-frequency distribution have been plotted in Fig. 3. The PWV kernel performs the worst. The variance of the ZAM distribution is still the highest, but this partially compensated for by the peak being large. The SPWV distribution appears to be the most stable, despite the fact that the spectrogram variance in noise is lower. The results show similar trends as in Fig. 1, even though these runs are for the signal plus noise case.

V. Conclusions

We have presented analytical expressions for the variance of an arbitrary time-frequency distribution for circular complex Gaussian stochastic processes. Using finite-extent discrete-time kernels, the Born-Jordan distribution has been found to have the lowest variance for a white-noise Gaussian process, given that the desirable t-f properties are satisfied except that of the frequency support. Its variance increases logarithmically without limit for improved frequency resolution, i.e., as more lags are included in the estimate. The kernel yielding minimum variance under all t-f constraints is derived using matrix representation and leads to a higher spectrum estimate variance than the pseudo Born–Jordan distribution.

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A Method for Improved Distribution Concentration in the Time-Frequency Analysis of Multicomponent Signals Using the L-Wigner Distribution

LJubiša Stanković

***Abstract*—The energy location in the Cohen class of time-frequency distributions is analyzed. If the instantaneous frequency is linear, then only the Wigner distribution produces the ideal energy concentration. The scaled version of the Wigner distribution (L-Wigner distribution), is used to improve the time-frequency representation of signals with nonlinear instantaneous frequencies. In the case of multicomponent signals, the cross terms, appearing in the Wigner distribution and in the L-Wigner distribution, can be easily removed or reduced in a computationally very efficient way. The theory is illustrated on the numerical examples with multicomponent noisy signals.**

I. Introduction

Time-frequency distributions have been intensively studied during the past decade. We refer to several review papers on the distributions

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for time-frequency analysis [1]–[4]. It is desirable that an energetic time-frequency distribution (TFD) of a signal $x(t)$ satisfies the following basic properties:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{TFD}_x(\omega,t)\,d\omega\,dt = E_x \tag{1}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{TFD}_x(\omega,t)\,d\omega = |x(t)|^2 \tag{2}$$

$$\int_{-\infty}^{\infty}\mathrm{TFD}_x(\omega,t)\,dt = |X(\omega)|^2 \tag{3}$$

where E_x and $X(\omega)$ denote the energy and Fourier transform of $x(t)$, respectively. Note that an infinite number of distributions satisfying (1)–(3) can be defined [1], [9]. Since the above properties do not tell anything about the local distribution of energy at a point (ω, t), we will impose some more specific requirements than the marginal ones given by (2) and (3).

Consider a complex signal $x(t)$

$$x(t) = r(t)e^{j\phi(t)} \tag{4}$$

with a slow-varying amplitude $r(t)$ compared to the phase $\phi(t)$ variations ($|r'(t)| \ll |\phi'(t)|$). The instantaneous frequency of $x(t)$ is defined by $\omega_i(t) = (d\phi(t)/dt) \equiv \phi'(t)$. The existence of $\phi'(t)$ is assumed. If the signal $x(t)$ is real, its analytic part, which can be written in form (4), will be used. For this form of signals, we will require that the ideal TFD has the instantaneous power $|r(t)|^2$ concentrated at the instantaneous frequency:

$$\mathrm{ITFD}(\omega,t) = 2\pi|r(t)|^2\,\delta[\omega-\phi'(t)]. \tag{5}$$

Additional details on the definition of ideal distribution may be found in [9].

In the section that follows, the commonly used distributions are compared with (5). The multicomponent signals and cross-term effects are studied in Section III. The analysis is illustrated on numerical examples.

II. Instantaneous Frequency Representation

A. Cohen Class of Time-Frequency Distributions

As it is known, all time-frequency (shift covariant) distributions satisfying the marginal properties belong to the general Cohen class of distributions (CD) [1], [2], [9]:

$$\mathrm{CD}(\omega,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(\theta,\tau)x(u+\tau/2) \cdot x^*(u-\tau/2)e^{-j\theta t - j\omega\tau + j\theta u}\,du\,d\theta\,d\tau \tag{6}$$

where $c(\theta,\tau)$ is an arbitrary kernel function. The marginal properties are satisfied if $c(\theta,0) = 1$ and $c(0,\tau) = 1$. The ideal distribution (5) may be easily translated into form (6) (taking the 2-D Fourier transform of ITFD(ω,t) and then its inverse 2-D Fourier transform) as

$$\mathrm{ITFD}(\omega,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r^2(u)e^{j\phi'(u)\tau} \cdot e^{-j\theta t - j\omega\tau + j\theta u}\,du\,d\theta\,d\tau. \tag{7}$$

Comparing (6) and (7), while having in mind the uniqueness of the Fourier transform, we get that the signal defined by (4) has the distribution equal to the ideal one iff

$$c(\theta,\tau)e^{j\phi(u+\tau/2) - j\phi(u-\tau/2)} = e^{j\phi'(u)\tau} \tag{8}$$

where $r(t)$ is treated as a constant inside the time interval determined by the kernel, i.e., $r(u+\tau/2)r(u-\tau/2)c(\theta,\tau) \simeq r^2(u)c(\theta,\tau)$.

If we restrict the analysis to the case of signal independent kernels (which is practically important), we get that the ideal distribution may be obtained only if $\phi^{(3)}(u) \equiv 0$ and $c(\theta,\tau) \equiv 1$. The previous requirements are met only by the Wigner distribution of the signal with a linear instantaneous frequency $\phi(t) = at^2/2 + bt$. For this signal, we get

$$\mathrm{CD}(\omega,t) = |r(t)|^2\int_{-\infty}^{\infty} c(-a\tau,\tau)e^{-j[\omega-\phi'(t)]\tau}\,d\tau. \tag{9}$$

If we assume that the instantaneous frequency $\phi'(t)$ is a constant, i.e., $\phi'(t) = b$, then substituting $a = 0$ into (9), it is easy to conclude that the CD is equal to the ITFD if $c(0,\tau) = 1$. This condition is fulfilled in many distributions from the Cohen class (generalized Wigner distribution, Rihaczek distribution, Pages distribution, Choi–Williams distribution ... [1], [6], [9], [13]).

B. L-Wigner Distribution

We have seen that the Wigner distribution is the only one from the Cohen class (with the signal independent kernels) having the ideal representation when the instantaneous frequency is a linear function. A way to improve the time-frequency representation of signals with a nonlinear instantaneous frequency is in the frequency linearization around a considered time instant t, provided that the value of instantaneous frequency is not changed. A distribution having these properties is the pseudo L-Wigner distribution (PLWD), which is introduced as

$$\mathrm{PLWD}(\omega,t) = \int_{-\infty}^{\infty} x^L\left(t+\frac{\tau}{2L}\right) x^{*L}\left(t-\frac{\tau}{2L}\right) w_L(\tau)e^{-j\omega\tau}\,d\tau \tag{10}$$

where $w_L(\tau)$ is a window, which is usually a real and even function, and L is any integer greater than 0. Taking $L = 1$ in (10), the pseudo Wigner distribution (PWD) is obtained.[1]

For the signals described by (4) and expanding $\phi(t+\tau/2L)$ and $\phi(t-\tau/2L)$ into a Taylor series, we get

$$\mathrm{PLWD}(\omega,t) = \frac{1}{2\pi}|r(t)|^{2L}\delta[\omega-\phi'(t)] *_\omega W_L(\omega) *_\omega \mathrm{FT}\Big\{\exp\Big[j\Big(\phi^{(3)}(t+\tau_1) + \phi^{(3)}(t-\tau_2)\Big)/\big(2^3L^{3-1}3!\big)\,\tau^3\Big]\Big\} \tag{11}$$

with the following notation: FT[ ] is the Fourier transform (FT) operator; $W(\omega) = \mathrm{FT}[w(t)]$; τ_1, τ_2 are variables in the interval $[0, \tau/2L]$; and $*_\omega$ is a convolution in ω.

The artifacts in the PLWD are due to the terms of third and higher order derivatives of the phase function (the odd ones). They are divided by the factor L^{n-1}. Values $L > 1$ produce a significant improvement of representation with respect to the PWD ($L = 1$). Note that for a large L, the PLWD approaches to the windowed ideal distribution for any frequency modulated signal.

C. L-Class of Time-Frequency Distributions

The idea of local linearization of the instantaneous frequency may be applied to the Cohen class of distributions, defining a generalized

[1] Recently, it has been shown that the L-Wigner distribution is a special, and in some ways, optimal case in the analysis of multicomponent signals using the multitime Wigner higher order distributions [14]. The properties of the L-Wigner distribution are analyzed in [10].

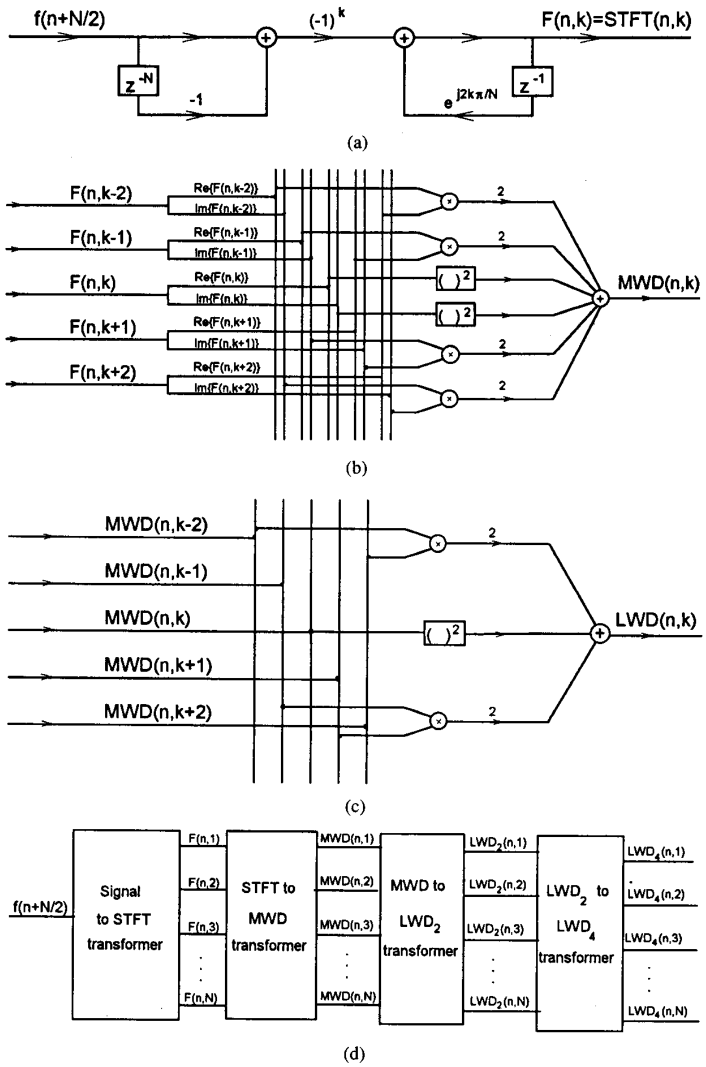

Fig. 1. On-line realization of the L-Wigner distribution: (a) Signal to STFT recursive transformer; (b) STFT to modified Wigner distribution transformer; (c) modified LWD (of order L) to modified LWD (of order $2L$) transformer; (d) complete system. The windows $w(\tau)$ and $P(\theta)$ are rectangular (with $L_d = 2$).

L-class of distributions:

$$\mathrm{LD}(\omega, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^L\left(u + \frac{\tau}{2L}\right) \cdot x^{*L}\left(u - \frac{\tau}{2L}\right) c_L(\theta, \tau) \cdot e^{-j\theta t - j\tau\omega + j\theta u}\, du\, d\tau\, d\theta. \tag{12}$$

Taking $c_L(\theta,\tau) = 1$, the L-Wigner distribution is obtained, while $c_L(\theta,\tau) = w_L(\tau)$ produces (10). The properties of the L-class of distributions, along with the specific distributions belonging to this class, are analyzed thoroughly in [16].

III. Multicomponent Signals

It is known that an annoying trait of the Wigner distribution is in the presence of cross terms [6], [7], [13]. The well-known cross-term effects may be even more emphatic in the PLWD because the Lth power of the signal introduces additional signal components. That is why the PLWD in form (10) may be useless for multicomponent signals. However, in the next analysis, we will show that the cross terms may be easily reduced or removed, preserving the appealing properties of the PLWD.

Let us consider multicomponent signals of the form

$$x(t) = \sum_{i=1}^{M} r_i(t) e^{j\phi_i(t)} \tag{13}$$

where $r_i(t)$ are the slow-varying amplitudes as in (4).

The basis for the analysis that follows is the short time Fourier transform (STFT), which is defined by [1]–[5]

$$\mathrm{STFT}(\omega, t) = \int_{-\infty}^{\infty} x(t+\tau) w(\tau) e^{-j\omega\tau}\, d\tau. \tag{14}$$

For the signal $x(t)$ given by (13), after the expansion of $\phi_i(t+\tau)$ into a Taylor series, assuming that the variations of $r_i(t+\tau)$ inside

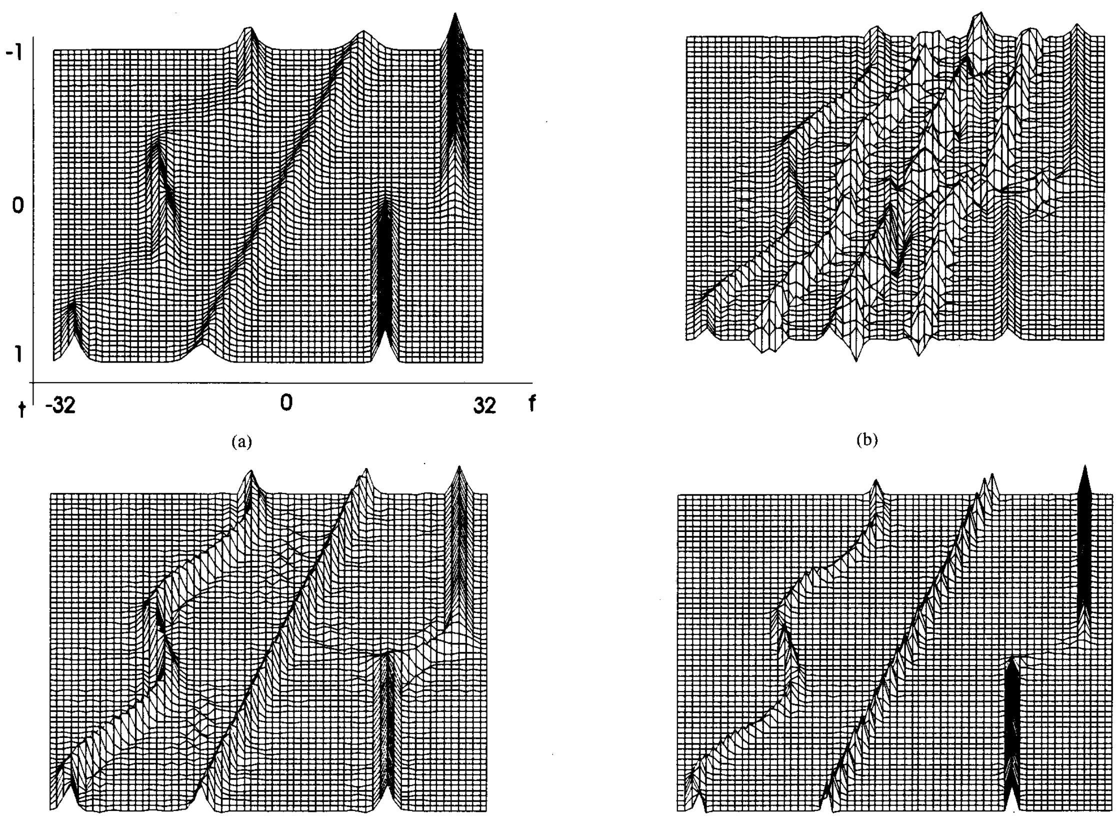

Fig. 2. Time-frequency representation of a multicomponent signal: (a) Spectrogram; (b) pseudo Wigner distribution; (c) modified pseudo Wigner distribution—MPWD; (d) modified pseudo L-Wigner distribution (MPLWD) with $L = 4$, with the Hanning window $w(\tau)$, whose width is $T = 1$; the rectangular window $P_d(i)$ whose width is $L_d = 3$; number of samples $N = 64$.

the window may be neglected ($r_i(t+\tau)w(\tau) \cong r_i(t)w(\tau)$), we get

$$\mathrm{STFT}(\omega,t) = \frac{1}{2\pi}\sum_{i=1}^{M} r_i(t)e^{j\phi_i^{(2)}(t)}\delta[\omega - \phi_i'(t)] *_\omega W(\omega) *_\omega FT\{e^{j\phi_i(t+\tau_1)\tau^2/2!}\}|. \tag{15}$$

From (15), we see that the absolute value of the STFT of a signal component is of the form similar to (5) convolved with the Fourier transform of the window and higher order terms of the derivatives of $\phi(t)$, starting with the second one.

From the STFT, which is given by (15), assuming that $\phi_i(t+\tau_1)$ is negligible inside the window, we get the spectrogram

$$\mathrm{SPEC}(\omega,t) = \sum_{i=1}^{M}\sum_{j=1}^{M} r_i(t)r_j(t)e^{j[\phi_i^{(2)}(t)-\phi_j(t)]} \cdot W[\omega - \phi_i'(t)]W^*[\omega - \phi_j'(t)]. \tag{16}$$

Suppose that the values of $W(\omega)$ may be considered to be zero for $|\omega| \geq W_B/2$ (where W_B is the width of $W(\omega)$ or the width of its main lobe). In that case, we can distinguish two cases:

1) If $\min[|\phi_i'(t) - \phi_j'(t)|] > W_B$ for all i, j and a given t, then the signal energy is concentrated only in the auto terms centered at the auto frequencies
2) If, for any l and $k|\phi_l'(t) - \phi_k'(t)| < W_B$, then between the instantaneous frequencies $\phi_l'(t)$ and $\phi_k'(t)$, energy of cross terms $r_l(t)e^{j\phi_l(t)}$ and $r_k(t)e^{j\phi_k(t)}$ exists.

Next, we will analyze the PWD ((10) with $L = 1$), which, for real windows, may be written using the STFT as

$$\mathrm{PWD}(\omega,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{STFT}(\omega+\theta,t) \cdot \mathrm{STFT}^*(\omega-\theta,t)\,d\theta. \tag{17}$$

For signals whose spectrogram is of form (16), we have

$$\mathrm{PWD}(\omega,t) = \frac{1}{\pi}\sum_{i=1}^{M}\sum_{j=1}^{M} r_i(t)r_j(t)e^{j[\phi_i(t)-\phi_j(t)]} \cdot \int_{-\infty}^{\infty} W[\omega+\theta-\phi_i'(t)] \cdot W^*[\omega-\theta-\phi_j'(t)]\,d\theta. \tag{18}$$

In the double summation in (18), the terms different from zero are those satisfying the following relations:

$$|\omega + \theta - \phi_i'(t)| < W_B/2$$

and

$$|\omega - \theta - \phi_j'(t)| < W_B/2. \tag{19}$$

Summing the previous inequalities, we get

$$\left|\omega - \frac{\phi_i'(t) + \phi_j'(t)}{2}\right| < W_B/2. \tag{20}$$

The Wigner distribution exists around the frequencies $\omega = (\phi_i'(t) + \phi_j'(t))/2$ for all i and j. It is evident that for $i \neq j$, the cross terms

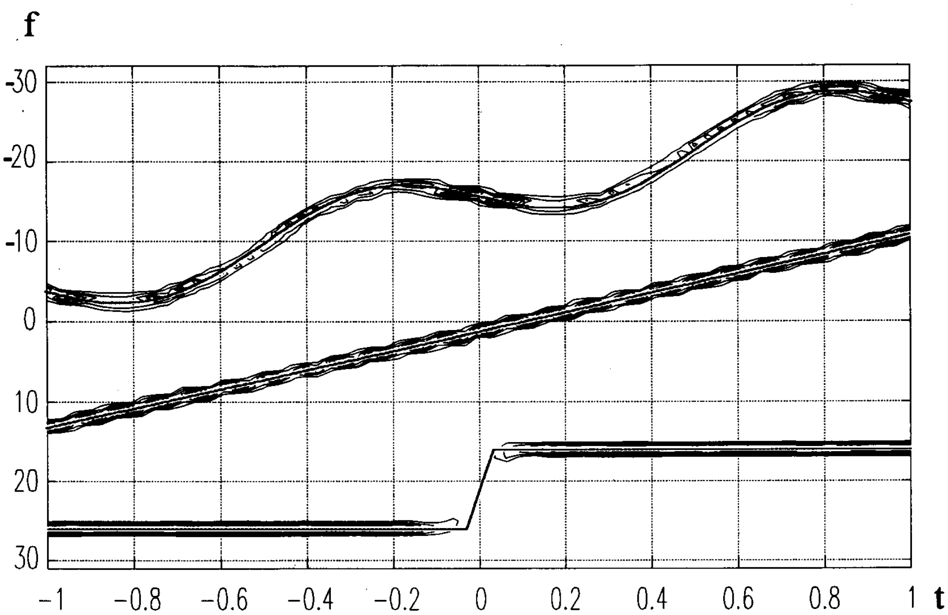

Fig. 3. Instantaneous frequencies obtained by the MPLWD with $L = 4$ (Contour graphics of Fig. 2(d)) along with the exact ones (solid lines in the middle of contours).

exist even if $\phi_i'(t)$ and $\phi_j'(t)$ are very far apart (in contrast to the spectrogram). Cross terms are centered between the instantaneous frequencies of the ith and jth signal components.

It is extremely interesting to investigate the location of the components contributing to the cross terms on the θ axis (18). From (19), we obtain

$$\left|\theta - \frac{\phi_i'(t) - \phi_j'(t)}{2}\right| < W_B/2. \tag{21}$$

The auto terms (for $i = j$) are obtained by the integration around $\theta = 0$ in the interval $|\theta| < W_B/2$, while the cross terms are obtained, integrating along the interval $|\theta - [\phi_i'(t) - \phi_j'(t)]/2| < W_B/2$. This is an interesting conclusion because we can eliminate cross terms between the components whose frequencies are further apart than $2W_B$. Using the window $P(\theta)$ in the integral (17), which will have the width W_P ($P(\theta) = 0$ for $|\theta| > W_P/2$) such that $W_B \le W_P < |\phi_i'(t) - \phi_j'(t)| - W_B$, the integration over auto terms will be performed completely (they will be the same as in the WD), and at the same time, the cross terms between the ith and jth components will be avoided. In that way, we arrived at the method for time frequency analysis

$$\begin{aligned}\text{MPWD}(\omega, t) = &\frac{1}{\pi}\int_{-\infty}^{\infty} P(\theta)\text{STFT}(\omega+\theta, t)\\ &\cdot \text{STFT}^*(\omega - \theta, t)\, d\theta\end{aligned} \tag{22}$$

which will preserve the appealing properties of the Wigner distribution but without cross terms. In [8] and [9], it has been shown that (22) may be realized in a numerically very efficient way.

Similarly, the L-Wigner distribution can be understood as a convolution of the Wigner distributions. For $L = 2$, the modified pseudo L-Wigner distribution (MPLWD) for cross-term elimination is in the form

$$\begin{aligned}\text{MPLWD}(\omega, t) = &\frac{1}{\pi}\int_{-\infty}^{\infty} P_1(\theta)\text{MPWD}(\omega+\theta, t)\\ &\cdot \text{MPWD}(\omega - \theta, t)\, d\theta\end{aligned} \tag{23}$$

where the properties of window $P_1(\theta)$ are the same as the properties of $P(\theta)$ in (22). Convolving two L-Wigner distributions with $L = 2$, we get the L -Wigner distribution with $L = 4$, and so on. That way, we can achieve the high distribution concentration, while at the same time avoiding the cross terms.

The discrete forms of (22) and (23), assuming that $P(\theta)$ is a rectangular window, are

$$\begin{aligned}\text{DMPWD}(k, n) = &\text{DSPEC}(k, n) + 2\sum_{i=1}^{Ld}\\ &\cdot Real\{\text{DSTFT}(k+i, n)\\ &\cdot \text{DSTFT}^*(k-i, n)\}\end{aligned} \tag{24}$$

where L_d is the width of the discrete form of $P(\theta)$, which will be denoted by $P_d(i)$.

The discrete Modified pseudo L-Wigner distribution (DMPLWD), with $L = 2$, is

$$\begin{aligned}\text{DMPLWD}(k, n) = &\text{DMPWD}(k, n) + 2\sum_{i=1}^{L_d}\\ &\cdot \{\text{DMPWD}(k+i, n)\\ &\cdot \text{DMPWD}(k-i, n)\}.\end{aligned} \tag{25}$$

For $L = 4, 8, \cdots$ the DMPLWD may be easily calculated repeating the procedure described by the last relation. A complete block diagram for on-line realization is shown in Fig. 1. Note that in this realization, the STFT is calculated using the recursive formula (details may be found in [5], [8], [9]):

$$\begin{aligned}\text{STFT}(k, n+1) = &\left[f\left(n+1+\frac{N}{2}\right) - f\left(n+1-\frac{N}{2}\right)\right]\\ &\cdot (-1)^k + \text{STFT}(k, n)e^{j(2\pi/N)k}.\end{aligned} \tag{26}$$

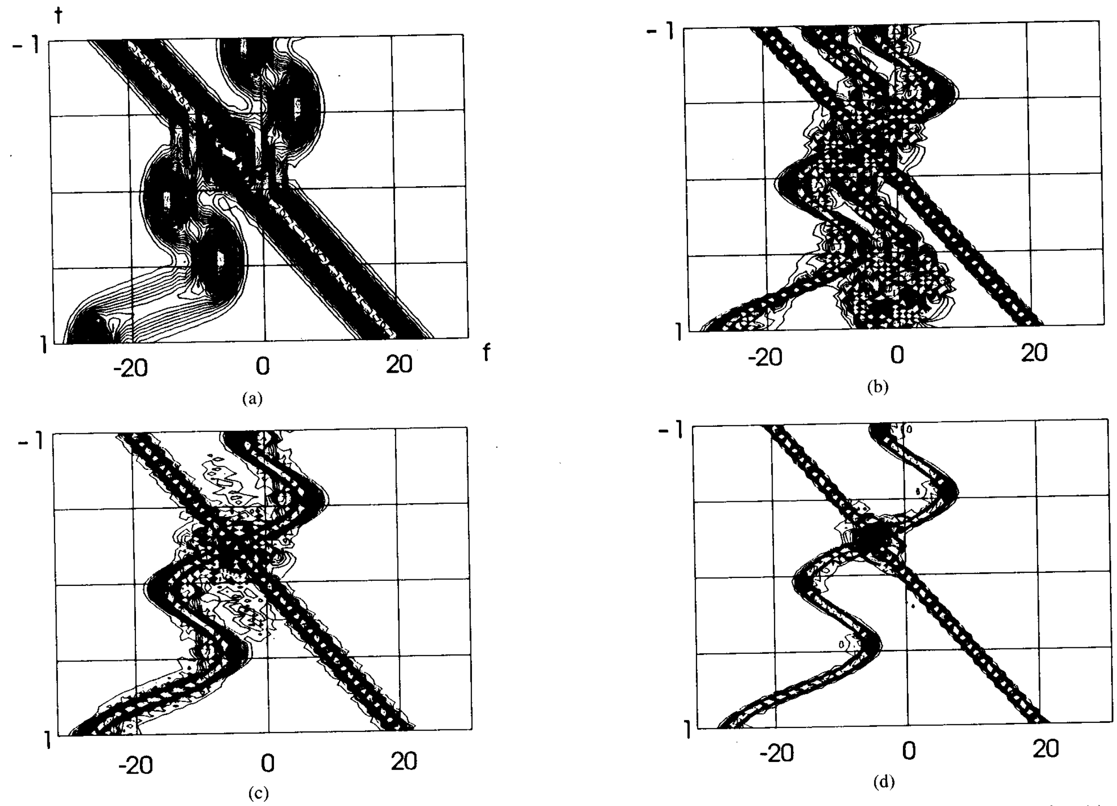

Fig. 4. Time-frequency representation of a multicomponent signal whose components intersect: (a) Spectrogram; (b) pseudo Wigner distribution; (c) modified pseudo Wigner distribution—MPWD; (d) modified pseudo L-Wigner distribution (MPLWD) with $L = 2$. With the windows as in Fig. 2.

IV. Decomposition of a Multicomponent Signal

The previous analysis may be used for a decomposition of $x(t)$, which is of the form

$$x(t) = \sum_{i=1}^{M} R_i e^{j\phi_i(t)} \tag{27}$$

where R_i are unknown constant amplitudes, and $\phi_i(t)$ are unknown phase functions, which are to be determined.

This problem may be resolved using the MPLWD. The only precondition that follows from the analysis of the MPLWD is that the instantaneous frequencies $\phi'_i(t)$ and $\phi'_j(t)$ (for any $i \neq j$) are apart at least $2W_B$ for all t in the considered time interval. Note that $\phi'_i(t_1) = \phi'_j(t_2)$ is allowed to be for different t_1 and t_2.

Using MPLWD ((10) and (11) calculated by (24)–(26)), we may determine $\phi'_i(t)$ for each i and t in the considered interval. The problem now becomes the well-known one of a signal $x(t)$ expansion on a set of known basis functions. Note that $\phi_i(t)$ may be obtained from the PLWD (i.e., from the $\phi'_i(t)$) within an integration constant C_i. This constant will be considered to be a part of R_i. Thus, we have the unknown complex coefficients $A_i = R_i e^{jC_i}$. The coefficients A_i will be determined, minimizing the mean-square error [5] from the set of linear equations

$$\langle x_1, x_i^*\rangle A_1 + \langle x_2, x_i^*\rangle A_2 + \cdots + \langle x_M, x_i^*\rangle A_M = \langle x(t), x_i^*\rangle \qquad i = 1, 2, \cdots, M \tag{28}$$

where $\langle x_i, x_j^*\rangle$ denotes the scalar product of x_i and x_j, with $x_i = \exp\left[\int_{t_1}^{t} \phi'_i(t)\, dt\right]$, and t_1 is any time instant in the considered interval.

V. Numerical Example

In the numerical example, a multicomponent signal of the form

$$x(t) = e^{-j12\pi(t-0.1)^2} + e^{j[-4\cos(2\pi t) - 12\pi(t+1.3)^2]} + e^{j[-10\pi t\, sign(\sin(\pi t/2)) + 42\pi t]} \tag{29}$$

is considered. We will add some noise; therefore, the signal $x(t) + n(t)$ will be used for calculations ($n(t)$ is a Gaussian white noise with SNR= 10 dB). The STFT (Fig. 2(a)) and the PWD (Fig. 2(b)) using a Hanning window are calculated, as well as the modified pseudo Wigner distribution (Fig. 2(c)) and Modified pseudo L-Wigner distribution (for $L = 2$ and $L = 4$ (Fig. 2(d)) with the same number of samples. The improvement of distribution concentration around the instantaneous frequencies, as well as the cross-term reduction (removal) using the modified pseudo L-Wigner distribution, is clearly shown in Fig. 2. The derivatives needed for signal decomposition, which is obtained by modified pseudo L-Wigner distribution ($L = 4$), as well as the exact ones, are given in Fig. 3. The coefficients obtained from system (28) are $A_1 = 0.9847e^{-j1.6510}$, $A_2 = 0.9939e^{-j1.0909}$, and $A_3 = 0.9702e^{j0.0223}$. The agreement with the exact ones ($A_1 = e^{-j1.6336}$, $A_2 = e^{-j1.1097}$, and $A_3 = 1$) is within the discretization and numerical integration error.

As a second example, we will consider the case when two signal components intersect, i.e., when it is not possible to satisfy the criterion for cross terms elimination for an instant t. The signal is taken in the form

$$x(t) = e^{j20\pi t^2} + e^{-j[8\sin(2\pi(t+1))+3\pi t(5+4t)]}. \tag{30}$$

The spectrogram, the pseudo Wigner distribution, the modified pseudo Wigner distribution (calculated using (24)), and the modified pseudo L-Wigner distribution (calculated using (25)) are given in Fig. 4(a)–(d), respectively. It is evident that the proposed method is efficient in cross-term removal anywhere in the time-frequency plane, except around the point of intersection, where they exist. That is exactly what we expected from the theoretical analysis presented in Section III.

VI. Conclusion

The energy concentration in the time-frequency plane using time-frequency distributions is analyzed. It is shown that if the instantaneous frequency is linear, only the Wigner distribution out of the Cohen class produces the ideal concentration. For the analysis of signals with nonlinear frequency, the L-Wigner distribution is used. A computationally efficient method for its implementation, without cross terms, is given.

Appendix A
Analysis of the Aliasing Effects

The analysis of aliasing effects is very interesting in the Wigner distribution because of its quadratic nature [11], [12]. In order to avoid aliasing, a signal has to be oversampled by a factor of 2 with respect to the sampling interval defined in the sampling theorem, or its analytic part has to be used. Here, we will show that the aliasing components appearing in the Wigner distribution may be eliminated in the same way as the cross terms.

Consider the discrete signal $x_d(t)$, which is formed by sampling a continuous signal $x(t)$.

$$x_d(t) = \sum_{k=-\infty}^{\infty} Tx(kT)\delta(t-kT) \tag{A1}$$

where T represents the sampling interval. The Fourier transform of $x_d(t)$ is a periodic function along the frequency axis with the period $\omega_p = 2\pi/T$ [5]

$$X_d(\omega) = \sum_{k=-\infty}^{\infty} X(\omega + k\omega_p). \tag{A2}$$

From (A2), we may conclude that $x_d(t)$ may be formally treated as a continuous multicomponent signal having an infinite number of components. The STFT for the sampled signal (4) is of the form

$$\mathrm{STFT}_d(\omega, nT) = r(t)e^{j\phi(t)} \sum_{k=-\infty}^{\infty} \cdot W(\omega + k\omega_p - \phi'(t))|_{t=nT} \tag{A3}$$

where we have neglected the distortions due to higher-order partial derivatives of the phase function.

Combining the previous relation with (17), the WD of discretized signal is obtained in the form

$$\mathrm{WD}_d(\omega, nT) = \frac{1}{\pi} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} |r(t)|^2 \cdot \int_{-\infty}^{\infty} W[\omega + \theta + k_1\omega_p - \phi'(t)] \cdot W^*[\omega - \theta + k_2\omega_p - \phi'(t)]\, d\theta|_{t=nT}. \tag{A4}$$

Using similar procedure as in the cross-term analysis, it may be seen that the integrand in (A4) is nonzero if the following holds:

$$-W_B/2 - (k_1 - k_2)\omega_p/2 < \theta < W_B/2 - (k_1 - k_2)\omega_p/2. \tag{A5}$$

The auto terms ($k_1 = k_2$) appear as a consequence of integration around the origin in the θ coordinate system. The closest aliasing components along θ axis are those for $k_1 - k_2 = \pm 1$. Obviously, they may be eliminated using a window $P(\theta)$, which is equal to zero along the θ axis for the values of θ outside the interval $|\theta| < \omega_p/2 - W_B/2$. Observe that this condition is usually significantly relaxed, as compared to the condition for eliminating cross terms (21). Thus, removal of cross terms by the modified Wigner distribution usually guarantees the elimination of the aliasing components. Therefore, the sampling in the MPWD may be done according to the sampling theorem. The same conclusions are valid for the MPLWD calculated from the MPWD.

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